Unit-1

Theory of Elasticity & Functional Approximating Methods:

Introduction to Theory of Elasticity: Definition of stress and strain – plane stress – plane strain – stress strain relations in three dimensional elasticity.

Introduction to Variational Calculus: Variational formulation in finite elements – Ritz method – Weighted residual methods – Galerkin – sub domain – method of least squares and collocation method - numerical problems

One Dimensional Problems: Discretization of domain, element shapes, discretization procedures, assembly of stiffness matrix, band width, node numbering, mesh generation, interpolation functions, local and global coordinates, convergence requirements, treatment of boundary conditions. Steady state heat transfer analysis : one dimensional analysis

TEXT BOOKS:

1. An introduction to Finite Element Method / JN Reddy / McGraw Hill
2. The Finite Element Methods in Engineering / SS Rao / Pergamon.

References

1. Tirupathi R. Chandrupatla and Ashok D. Belugundu (2011) Introdution to Finite Elements in Engineering, Prentice Hall.
2. Seshu P., Text Book of Finite Element Analysis, Prentice Hall, New Delhi, 2007.
3. Zienkiewicz O.C., Taylor R.L., Zhu J.Z. (2011), The Finite Element Method: Its basis and fundamentals, Butterworth Heinmann.

E-RESOURCES: **https://nptel.ac.in/courses/112/104/112104193/**
https://mecheng.iisc.ac.in/suresh/me237/feaNotes

Unit-2

Analysis of Trusses: Finite element modelling, coordinates and shape functions, assembly of global stiffness matrix and load vector, finite element equations, treatment of boundary conditions, stress, strain and support reaction calculations.

Analysis of Beams: Element stiffness matrix for Hermite beam element, derivation of load vector for concentrated and UDL, simple problems on beams.

Unit-3

Two Dimensional Problems: Finite element modelling of two dimensional stress analysis with constant strain triangles **CST** and treatment of boundary conditions,

Higher order and isoparametric elements: Two dimensional four noded isoparametric elements and numerical integration.

Axisymmetric Problems: Formulation of axisymmetric problems.

Unit-4

Dynamic Analysis: Formulation of finite element model, element consistent and lumped mass matrices, evaluation of eigen values and eigen vectors, free vibration analysis.Steady state heat transfer analysis: one dimensional analysis of a fin.

Introduction to FE software.

FEM

- Numerical method for engineering solution.

Finite

- Reduces the degrees of freedom from infinite to finite

Element

- All of the calculations are made at a limited number of points known as nodes.
- The entity joining nodes and forming a specific shape such as quadrilateral or triangular is known as an Element.

Method

“”FEM is a numerical technique to find the approximate solutions of partial differential equations. It was originated from the need of solving complex elasticity and structural analysis problems in Civil, Mechanical and Aerospace engineering.””

Difference between FEM and FEA ??

Finite Element Method (**FEM**) involves complex mathematical procedures (like a theory manual, lots of equations and mathematics).

Finite Element Analysis (**FEA**) involves applying **FEM** to solve real world/ engineering problems.

https://www.amazon.com/DonaldsPractical-Stress-Analysis-Elements-Hardcover-dp-B005KZ2UJE/dp/B005KZ2UJE/ref=mt_other?_encoding=UTF8&me=&qid=

In a structural simulation, FEM helps in producing stiffness & strength visualizations. It also helps to minimize material weight and its cost of the structures. FEM allows for detailed visualization and indicates the distribution of stresses and strains inside the body of a structure.

In structural analysis, FEM helps in producing stiffness and strength visualizations. It helps in minimising the material and cost of the structures.

Modern FEM packages (ansys, abaqus), include specific components such as Fluid, thermal, EM and structural working environments

Several modern FEM packages include specific components such as fluid, thermal, electromagnetic, and structural working environments. FEM allows entire designs to be constructed refined and optimized before the design is manufactured.

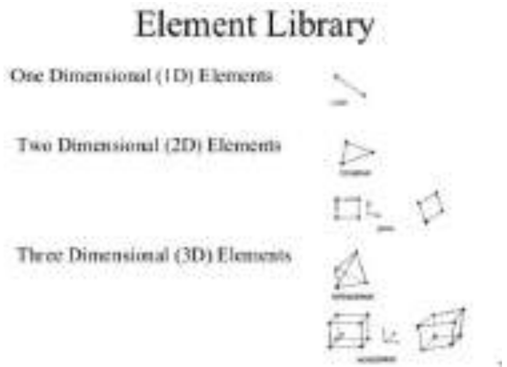
Numerical Methods:

Numerical Methods which are commonly used to solve solid and fluid mechanics problems are given below.

1. Finite Difference Method.
2. Finite Volume Method.
3. Finite Element Method.
4. Boundary Element Method.
5. Meshless Method.

→ Concepts of elements and Nodes:

Any Continuum/Domain can be divided into number of pieces with very small dimensions. These small pieces/sub domains of finite dimension are called "Finite Elements". These Elements are connected through a number of joints which are called 'Nodes'. While discretizing the structural system, it is assumed that the elements are attached to the adjacent elements only at the nodal points. Each element contains the material and geometrical properties. The material properties inside an element are assumed to be constant. The elements may be 1D, 2D or 3D elements.



* Stress : The Restoring force per unit area setup inside the body is known as stress.

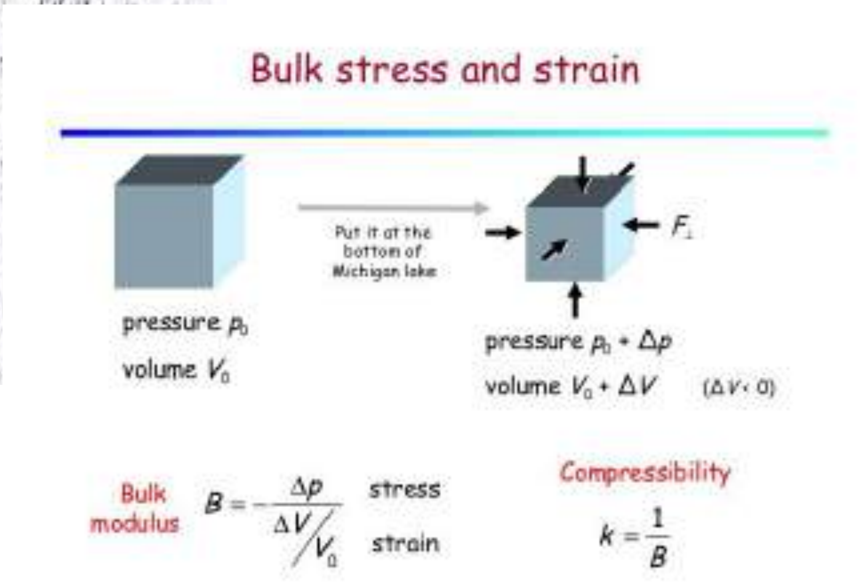
$$\text{Stress} = \frac{\text{Restoring force}}{\text{Area}} = \frac{F}{A}$$

Units : N/m^2

Types of stresses:

Stress

- Normal stress (Normal to Plain Surface)
- Volume or Bulk stress
- Tangential or Shear stress



Bulk or **hydrostatic stress**, also known as volumetric stress is a component of stress which contains uniaxial stresses, but not shear stresses. A specialized case of hydrostatic stress, contains isotropic compressive stress, **which changes only in volume,** but not in shape.

$$\sigma_h = \frac{I_i}{3} = \frac{\sigma_{xx} + \sigma_{yy} + \sigma_{zz}}{3}$$

Hydrostatic stress is equivalent to the average of the uniaxial stresses along three orthogonal axes

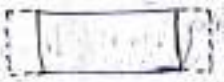
*** Strain:** The ratio of change in dimension by Original dimension is known as strain.

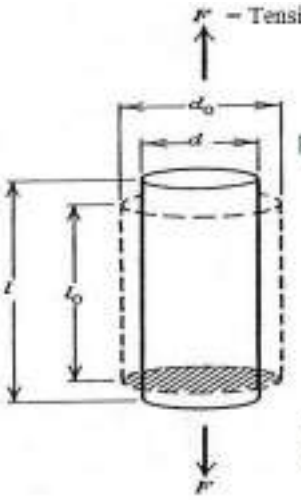
$$\text{Strain} = \frac{\text{change in dimension}}{\text{Original dimension}} = \frac{\Delta L}{L} \quad \text{(No Units)}$$

Types of strains:

Strain

- Normal strain → Size change
- Volume or Bulk strain: $\varepsilon_v = \frac{\Delta V}{V}$
- Shear strain → Shape change





$$\text{Poisson Ratio} = \frac{\text{Lateral Strain}}{\text{Linear Strain}}$$

$$\text{Linear strain} = \frac{\Delta l}{l} = \frac{l - l_o}{l_o}$$

$$\text{Lateral Strain} = \frac{\Delta d}{d_o} = \frac{d - d_o}{d_o}$$

Poisson's ratio values
0 cork, 0.3 MS, 0.5 Rubber
Limits 0 to .5

Eps v= Eps x+Eps y+Eps z

For cylinder-----------?
And sphere ………..?

★ Relation between Stress, Strain & Shear Stress & Shear Strain:

(1) Normal

$$\sigma \propto \epsilon$$

$$\sigma = E\epsilon$$

Here E = Modulus of elasticity or Young's Modulus.

σ = Stress (Normal), ϵ = Normal Strain.

(2) Shear

$$\tau \propto \gamma$$

$$\tau = G\gamma$$

Here τ = Shear stress, γ = Shear strain

G = Shear Modulus or Modulus of Rigidity

(3) Bulk or Volume

$$\sigma_v \propto \epsilon_v$$

$$\sigma_v = K\epsilon_v$$

Here σ_v = Bulk stress, ϵ_v = Bulk strain

K = Bulk Modulus.

So, Here E, G, K, & μ (Poisson's Ratio) are called elastic constants because, the values of the above doesn't change with the change in size or shape, but changes with the material.

Equilibrium of an Elastic Body : Consider a body with surface '*S*' occupying a volume '*V*', in which points are located by a coordinate system (x, y and z coordinates) shown in figure (1). At particular part, boundary is constrained. Some other region of boundary is subjected to traction '*T*' (uniformly distributed force or load per unit area). Deformation of the body occurs due to force and is specified at constrained part of the boundary. Displacement of any point $x = (x, y, z)^T$ is given by $u = (u, v, w)^T$.

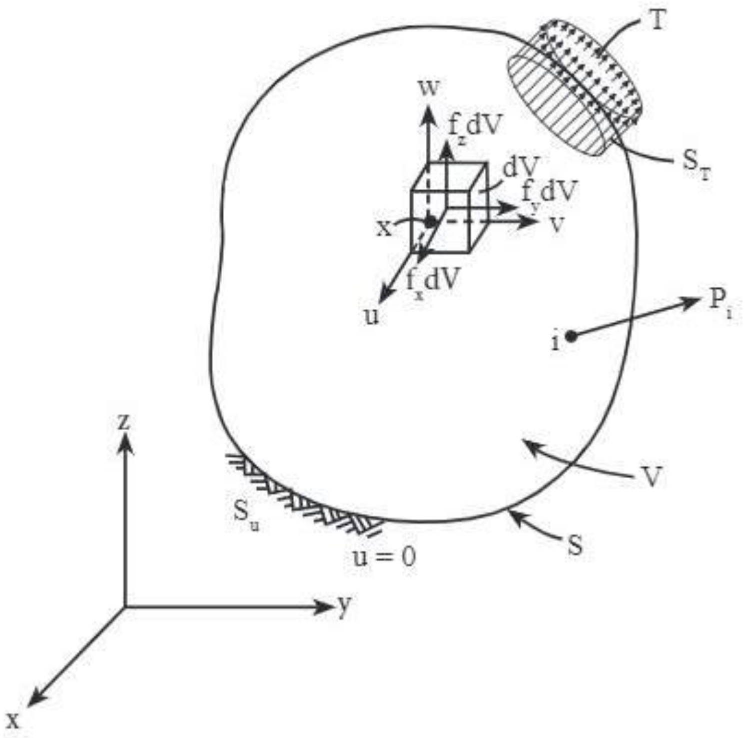
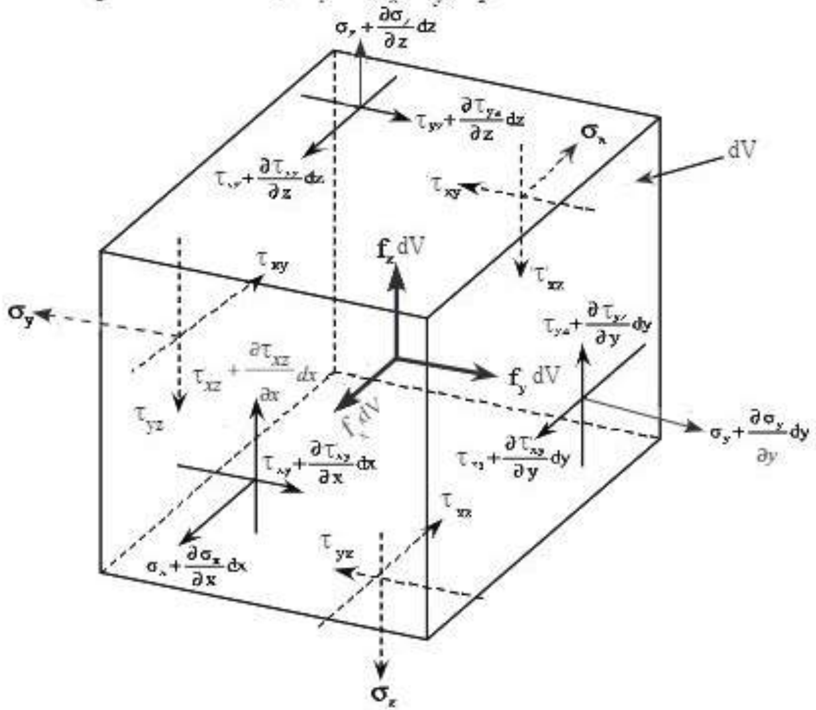


Figure (1)

Types of forces acting on the body are surface loads (friction, viscous drag) which exists whenever one body moves past to other body in contact $T = (T_x, T_y, T_z)^T$, body loads (forces distributed on volume of body like self weight, inertia, centrifugal forces, temperature, etc.) $f = (f_x, f_y, f_z)^T$ and point loads (loads concentrated on a point in continuum like tensile or compressive loads) $P_i = (P_x, P_y, P_z)^T$.



$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + f_z = 0 \qquad \dots (3)$$

(ii) Strain-displacement Relations

Considering the deformation of the *dx-dy* face,

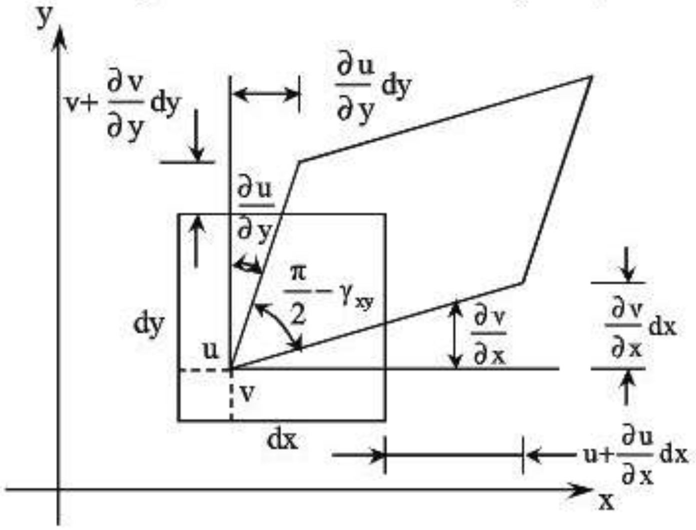


Figure (3): Deformed Elemental Surface

Then, corresponding strains are given by,

$$\varepsilon = [\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{yz}, \gamma_{xz}, \gamma_{xy}]^T$$

Where,

$\varepsilon_x, \varepsilon_y, \varepsilon_z$ – Normal strains

$\gamma_{yz}, \gamma_{xz}, \gamma_{xy}$ – Engineering shear strains

Figure (2): Equilibrium of Elemental Volume

There are three sets of equations of equilibrium in theory of elasticity, they are,

(i) Differential equations of equilibrium.

(ii) Strain-displacement relations.

(iii) Stress-strain relations.

(I) **Differential Equations of Equilibrium**

For elemental volume dV, 3×3 symmetric matrix is used to specify the components of the stress tensor. But for instant six independent components are employed to represent the stress.

$$\sigma = [\sigma_x, \sigma_y, \sigma_z, \tau_{yz}, \tau_{xz}, \tau_{xy}]^T$$

Where,

$\sigma_x, \sigma_y, \sigma_z$ – Normal stresses

$\tau_{yz}, \tau_{xz}, \tau_{xy}$ – Shear stresses

Considering equilibrium of an elemental volume, $\Sigma f_x = 0$, $\Sigma f_y = 0$, $\Sigma f_z = 0$ and considering $dV = dxdydz$, and by using, forces are obtained by multiplying stresses with their corresponding areas, equilibrium equations are given by,

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + f_x = 0 \quad \text{... (1)}$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + f_y = 0 \quad \text{... (2)}$$

Considering all the faces for small deformations,

$$\varepsilon = \left[\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial w}{\partial z}, \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}, \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right]^T$$

(iii) **Stress-strain Relations**

Applying Hooke's law to elemental volume 'dV', strains in terms of stress and material properties of isotropic materials (elastic modulus and Poisson's ratio) are obtained as,

$$\varepsilon_x = \frac{\sigma_x}{E} - \frac{\sigma_y}{E}\mu - \frac{\sigma_z}{E}\mu \quad \text{...(4)}$$

$$\varepsilon_y = -\frac{\sigma_x}{E}\mu + \frac{\sigma_y}{E} - \frac{\sigma_z}{E}\mu \quad \text{...(5)}$$

$$\varepsilon_z = -\frac{\sigma_x}{E}\mu - \frac{\sigma_y}{E}\mu + \frac{\sigma_z}{E} \quad \text{...(6)}$$

$$\gamma_{yz} = \frac{\tau_{yz}}{G}$$

$$\gamma_{xz} = \frac{\tau_{xz}}{G}$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G}$$

Where,

E – Young's modulus

μ – Possion's ratio

G – Modulus of rigidity

And,

$$G = \frac{E}{2(1+\mu)}$$

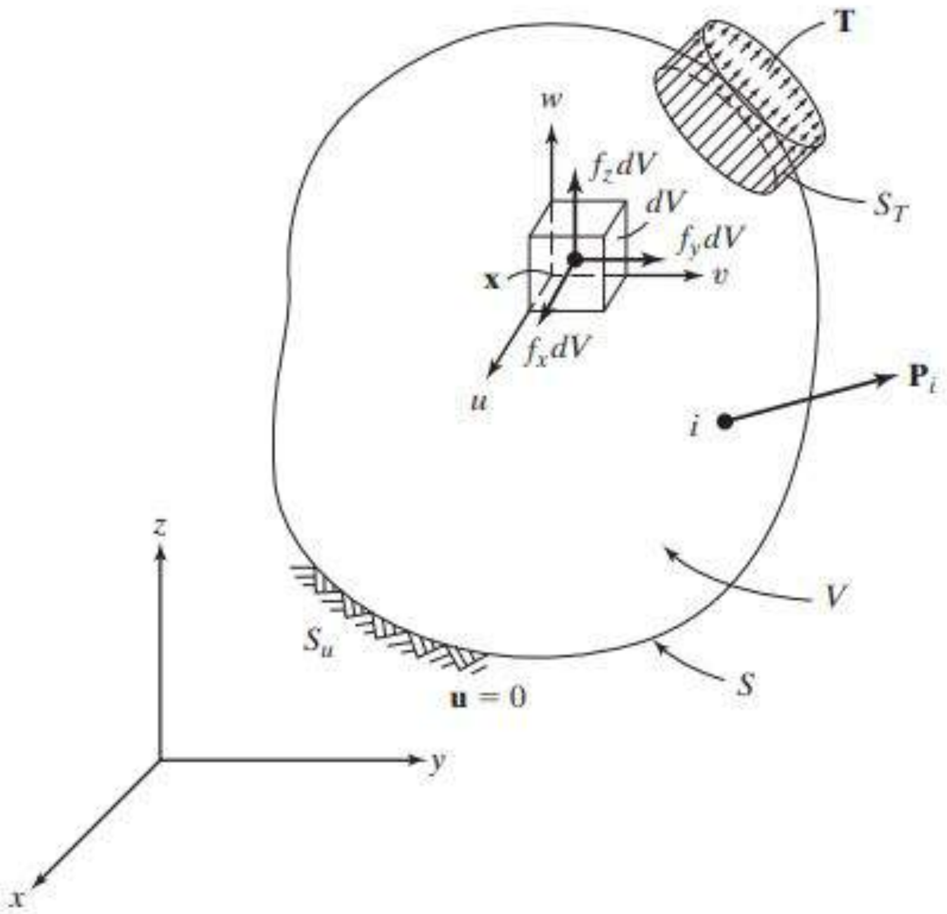


FIGURE 1.1 Three-dimensional body.

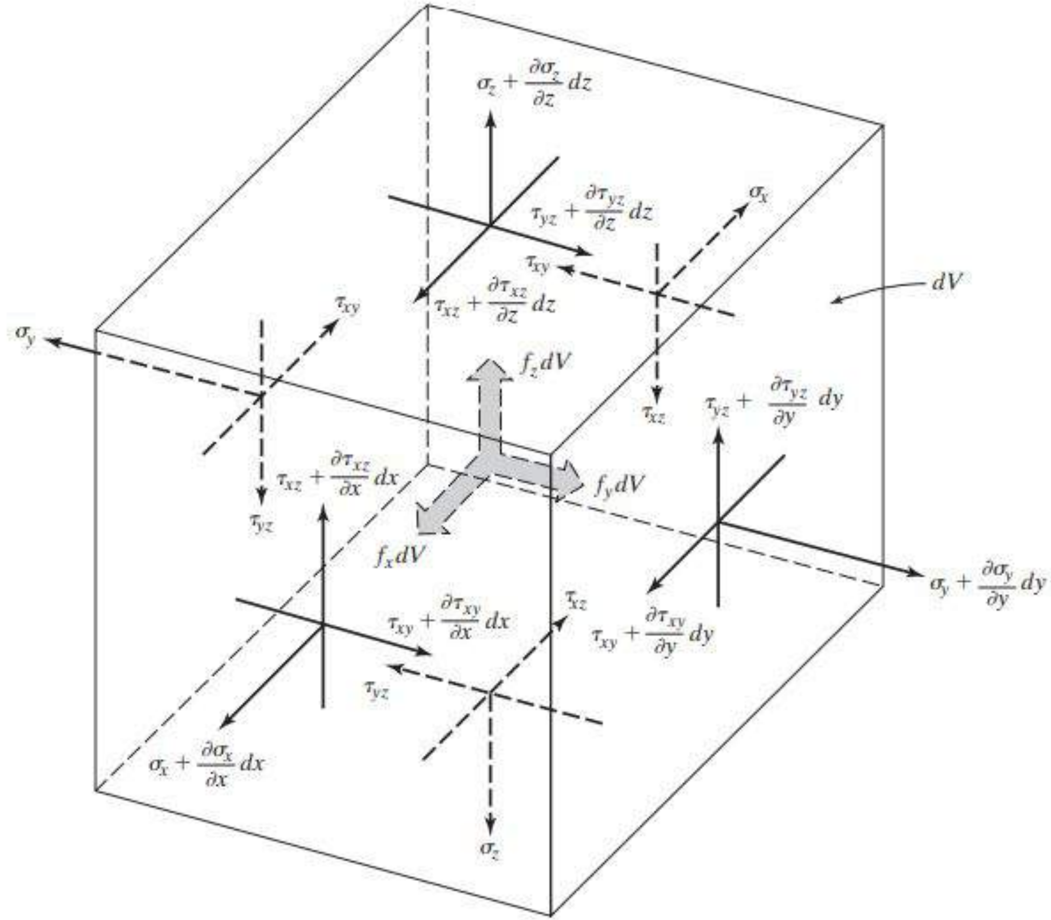


FIGURE 1.2 Equilibrium of elemental volume.

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + f_x = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + f_y = 0$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + f_z = 0$$

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$

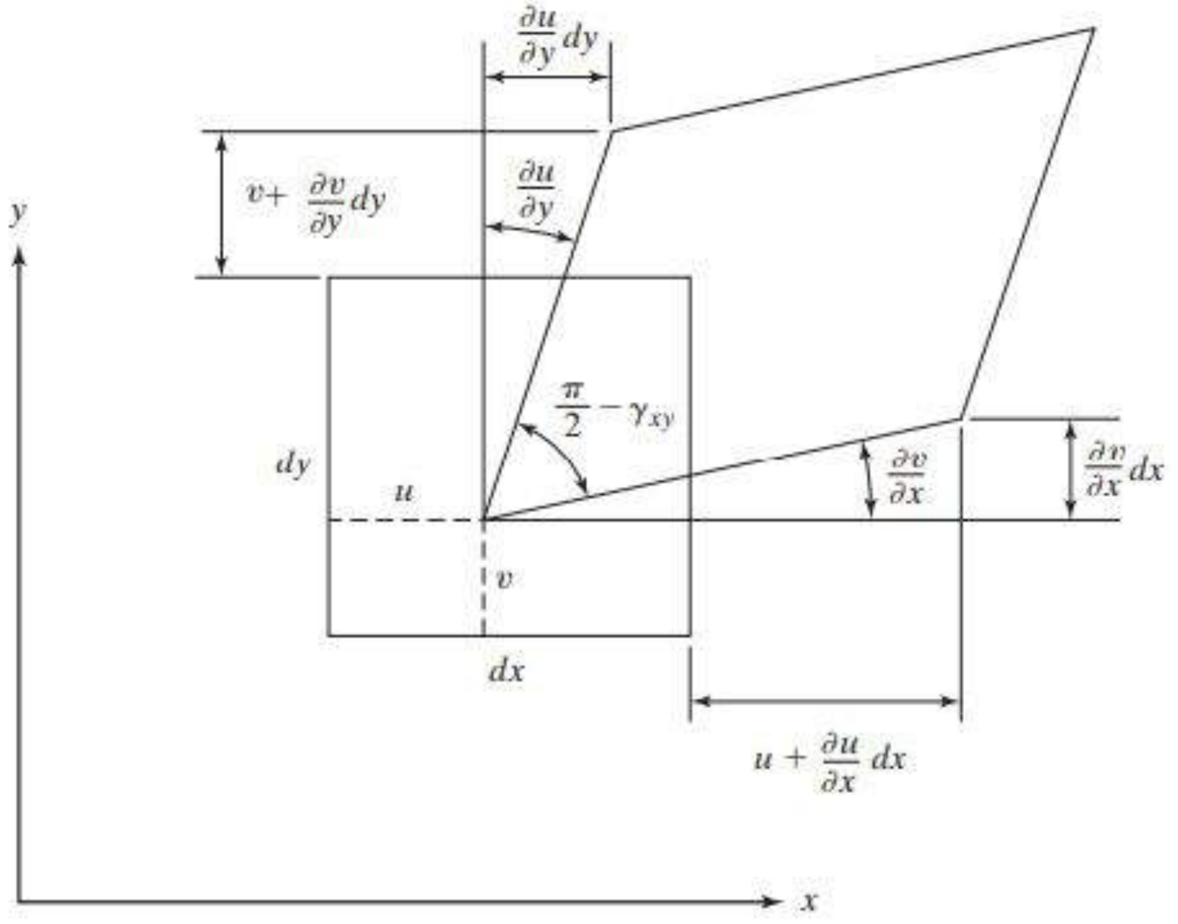


FIGURE 1.4 Deformed elemental surface.

$$\boldsymbol{\epsilon} = \left[\epsilon_x, \epsilon_y, \epsilon_z, \gamma_{yz}, \gamma_{xz}, \gamma_{xy}\right]^{\mathrm{T}}$$

$$\boldsymbol{\epsilon} = \left[\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial w}{\partial z}, \frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}, \frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}, \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right]^{\mathrm{T}}$$

$$\varepsilon_x = \frac{\sigma_x}{E} - \frac{\sigma_y}{E}\mu - \frac{\sigma_z}{E}\mu$$

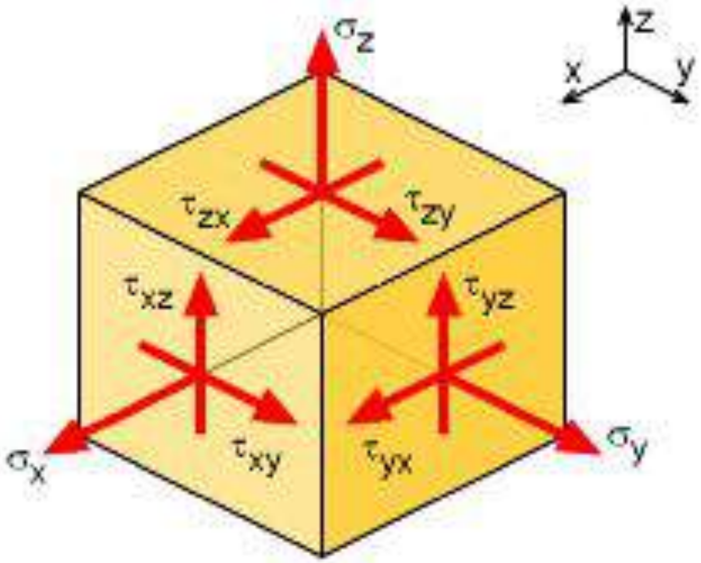
$$\varepsilon_y = -\frac{\sigma_x}{E}\mu + \frac{\sigma_y}{E} - \frac{\sigma_z}{E}\mu$$

$$\varepsilon_z = -\frac{\sigma_x}{E}\mu - \frac{\sigma_y}{E}\mu + \frac{\sigma_z}{E}$$

$$\gamma_{yz} = \frac{\tau_{yz}}{G}$$

$$\gamma_{xz} = \frac{\tau_{xz}}{G}$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G}$$



$$\varepsilon_x = \frac{\sigma_x}{E} - \frac{\sigma_y}{E}\mu - \frac{\sigma_z}{E}\mu$$

$$\varepsilon_y = -\frac{\sigma_x}{E}\mu + \frac{\sigma_y}{E} - \frac{\sigma_z}{E}\mu$$

$$\varepsilon_z = -\frac{\sigma_x}{E}\mu - \frac{\sigma_y}{E}\mu + \frac{\sigma_z}{E}$$

$$\epsilon_x + \epsilon_y + \epsilon_z = \frac{(1 - 2v)}{E}(\sigma_x + \sigma_y + \sigma_z)$$

$$\varepsilon_x = \frac{\sigma_x}{E} - \frac{\sigma_y}{E}\mu - \frac{\sigma_z}{E}\mu$$

$$\varepsilon_y = -\frac{\sigma_x}{E}\mu + \frac{\sigma_y}{E} - \frac{\sigma_z}{E}\mu$$

$$\varepsilon_z = -\frac{\sigma_x}{E}\mu - \frac{\sigma_y}{E}\mu + \frac{\sigma_z}{E}$$

$$\gamma_{yz} = \frac{\tau_{yz}}{G}$$

$$\gamma_{xz} = \frac{\tau_{xz}}{G}$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G}$$

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{bmatrix}$$

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{(1-2\nu)}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{(1-2\nu)}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{(1-2\nu)}{2} \end{bmatrix}}_{} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{bmatrix}$$

D = Constitutive matrix

Special cases

→ **One Dimension :**

In One dimension, we have normal stress σ along x and and the corresponding normal strain i.e. ϵ.

Stress - strain relation, $\sigma = E\epsilon$

→ **Two Dimensions :**

In two dimensions, the problems are modeled as plane stress & Plane strain.

Plane stress :

A state of Plane stress is said to exist when the elastic body is very thin and there are no loads applied in the coordinate direction parallel to the thickness. In other words, for some two dimensional objects the stresses can be produced only in two directions and not possible in the third direction.

Plane stress analysis includes problems such as Plates with holes, fillets or other changes in geometry.

Special cases

$$\varepsilon_x = \frac{\sigma_x}{E} - \frac{\sigma_y}{E}\mu - \frac{\sigma_z}{E}\mu$$

$$\varepsilon_y = -\frac{\sigma_x}{E}\mu + \frac{\sigma_y}{E} - \frac{\sigma_z}{E}\mu$$

$$\varepsilon_z = -\frac{\sigma_x}{E}\mu - \frac{\sigma_y}{E}\mu + \frac{\sigma_z}{E}$$

$$\gamma_{yz} = \frac{\tau_{yz}}{G}$$

$$\gamma_{xz} = \frac{\tau_{xz}}{G}$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G}$$

Strain in X-direction:

$$\epsilon_x = \frac{\sigma_x}{E} - \vartheta\frac{\sigma_y}{E}$$

Strain in Y-direction: & Z-direction

$$\epsilon_y = -\vartheta\frac{\sigma_x}{E} + \frac{\sigma_y}{E} \qquad \epsilon_z = -\vartheta\frac{\sigma_x}{E} - \vartheta\frac{\sigma_y}{E} + 0$$

& We have

$$\tau_{xy} \propto \gamma_{xy}$$

$$\tau_{xy} = G\gamma_{xy}$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G}$$

$$\gamma_{xy} = \tau_{xy} \times \frac{1}{\left(\frac{E}{2(1+\vartheta)}\right)}$$

$$\gamma_{xy} = \frac{2(1+\vartheta)}{E}\tau_{xy} \qquad \because G = \frac{E}{2(1+\vartheta)}$$

Now Coming to the Stress - Strain Relation for plane stress

$$\epsilon_x = \frac{\sigma_x}{E} - \vartheta\frac{\sigma_y}{E}$$

$$\epsilon_y = -\vartheta\frac{\sigma_x}{E} + \frac{\sigma_y}{E}$$

$$\epsilon_z = -\vartheta\frac{\sigma_x}{E} - \vartheta\frac{\sigma_y}{E}$$

$$\gamma_{xy} = \frac{2(1+\vartheta)}{E}\tau_{xy}$$

Special cases

Formation of Matrix

$$\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & 0 \\ 0 & 0 & \frac{2(1+\nu)}{E} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

$$\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & 0 \\ 0 & 0 & \frac{2(1+\nu)}{E} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}$$

$$\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}$$

$$\therefore \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = E \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix}^{-1} \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix}$$

Let $A = \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix}$

$$|A| = 2(1+\nu) + \nu\left(-2\nu(1+\nu)\right)$$

$$= (1+\nu)2\left[1-\nu^2\right]$$

$$= 2(1+\nu)(1-\nu^2)$$

$$\text{Cofactor of } A = \begin{bmatrix} 2(1+\nu) & 2\nu(1+\nu) & 0 \\ 2\nu(1+\nu) & 2(1+\nu) & 0 \\ 0 & 0 & (1-\nu^2) \end{bmatrix}$$

$$= 2(1+\nu) \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{(1-\nu)}{2} \end{bmatrix}$$

$$\therefore A^{-1} = \frac{C^T}{|A|} = \frac{\cancel{2(1+\nu)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{(1-\nu)}{2} \end{bmatrix}}{\cancel{2(1+\nu)}(1-\nu^2)}$$

Special cases

$$A^{-1} = \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

$$\therefore \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

$$\sigma = D\epsilon$$

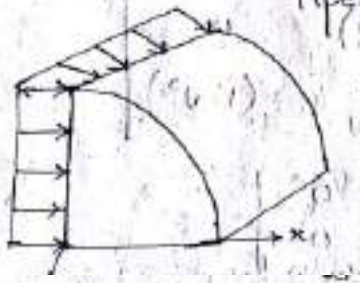
D = Stress - strain Relationship.

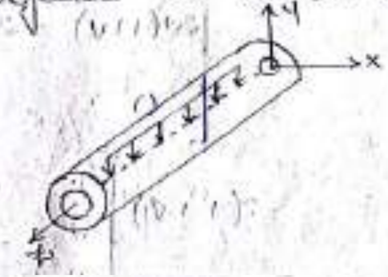
$$\Rightarrow D = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

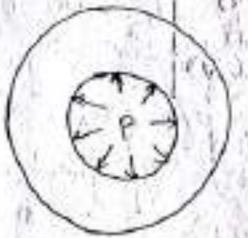
Plane Strain :

The state of plane strain occurs in members that are not free to expand in the direction perpendicular to the plane of applied forces / loads. Plane strain condition refers the occurrence of strain in the body in two directions only and in the third direction the strain is negligible & equal to Zero.

Examples :- Dams subjected to horizontal loading.
Pipes subjected to verticle loading.







$\epsilon_z = 0$
$\gamma_{yz} = 0$
$\gamma_{zx} = 0$

Here

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} (1-\nu) & \nu & 0 \\ \nu & (1-\nu) & 0 \\ 0 & 0 & \left(\frac{1-2\nu}{2}\right) \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

$$\{\sigma\} = [D]\{\epsilon\}$$

$$\frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} = \text{Constitutive Matrix for Plane-strain condition.}$$

Plane Strain :

The state of plane strain occurs in members that are not free to expand in the direction perpendicular to the plane of applied forces/loads. Plane strain condition refers the occurrence of strain in the body in two directions only and in the third direction the strain is negligible & equal to zero.

$$
\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{bmatrix}
= \frac{E}{(1+\nu)(1-2\nu)}
\begin{bmatrix}
1-\nu & \nu & \nu & 0 & 0 & 0 \\
\nu & 1-\nu & \nu & 0 & 0 & 0 \\
\nu & \nu & 1-\nu & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{(1-2\nu)}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{(1-2\nu)}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{(1-2\nu)}{2}
\end{bmatrix}
\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{bmatrix}
$$

For 3 D problems we have, $\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\epsilon}$

$$\boldsymbol{\sigma} = [\sigma_x, \sigma_y, \sigma_z, \tau_{yz}, \tau_{xz}, \tau_{xy}]^{\mathrm{T}}$$

$$\boldsymbol{\epsilon} = [\epsilon_x, \epsilon_y, \epsilon_z, \gamma_{yz}, \gamma_{xz}, \gamma_{xy}]^{\mathrm{T}}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1+v)(1-2v)} \begin{bmatrix} 1-v & v & v & 0 & 0 & 0 \\ v & 1-v & v & 0 & 0 & 0 \\ v & v & 1-v & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5-v & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5-v & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5-v \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix}$$

Plane Strain

$$\epsilon_z = 0$$
$$\gamma_{yz} = 0$$
$$\gamma_{xz} = 0$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1+v)(1-2v)} \begin{bmatrix} 1-v & v & v & 0 & 0 & 0 \\ v & 1-v & v & 0 & 0 & 0 \\ v & v & 1-v & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5-v & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5-v & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5-v \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix}$$

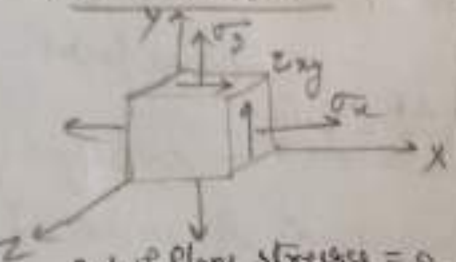
$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1+v)(1-2v)} \begin{bmatrix} 1-v & v & 0 \\ v & 1-v & 0 \\ 0 & 0 & 0.5-v \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [\mathbf{D}] \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

Plane stress: $$\mathbf{D} = \frac{E}{1-\nu^2}\begin{pmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{pmatrix}$$

Plane strain: $$\mathbf{D} = \frac{E}{(1+\nu)(1-2\nu)}\begin{pmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{pmatrix}$$

Plane stress



Out of Plane stresses = 0

→ stresses related to z are zero

$\sigma_{zz}, \tau_{xz}, \tau_{yz} = 0$

ϵ_{zz} non zero

$\gamma_{xz} = 0, \gamma_{yz} = 0$

shear strain related to z direcⁿ = 0.

only in plane stresses

Thin body

Plane strain

Out of plane strains = 0

strains related to z are 0

$\epsilon_{zz}, \gamma_{xz}, \gamma_{yz} = 0$

σ_z non zero

$\tau_{xz}, \tau_{yz} = 0$

shear stresses related to z are zero

Thick body

$$\epsilon_z = 0 = \frac{\sigma_z}{E} - \nu\left(\frac{\sigma_x + \sigma_y}{E}\right)$$

$$\boxed{\sigma_z = \nu(\sigma_x + \sigma_y)}$$

1.4. If a displacement field is described by

$$u = (x^2 + 4y^2 - 16xy)10^{-4}$$
$$v = (y^2 - 5x + 8y)10^{-4}$$

determine ϵ_x, ϵ_y, γ_{xy} at the point $x = 1, y = 0$.

1.3. In a plane strain problem, we have

$$\sigma_x = 30{,}000 \text{ psi}, \sigma_y = -15{,}000 \text{ psi}$$
$$E = 30 \times 10^6 \text{ psi}, \nu = 0.3$$

Determine the value of the stress σ_z.

1.6. A displacement field

$$u = 2 + 2x + 4x^2 + 3xy^2$$
$$v = xy - 8x^2$$

is imposed on the square element shown in Fig. P1.6.

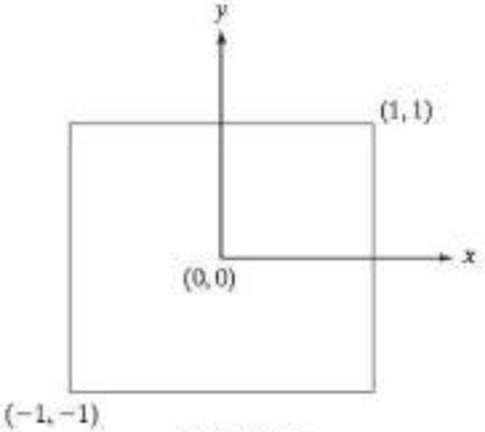


FIGURE P1.6

(a) Write down the expressions for ϵ_x, ϵ_y, and γ_{xy}.
(b) Plot contours of ϵ_x, ϵ_y, and γ_{xy} using, say, MATLAB software.
(c) Find where ϵ_x is maximum within the square.

1.3 Plane strain condition implies that

$$\varepsilon_z = 0 = -\nu\frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} + \frac{\sigma_z}{E}$$

which gives

$$\sigma_z = \nu(\sigma_x + \sigma_y)$$

We have, $\sigma_x = 20000$ psi $\quad \sigma_y = -10000$ psi $\quad E = 30\times10^6$ psi $\quad \nu = 0.3$.

On substituting the values,

$$\sigma_z = 3000 \text{ psi} \qquad \blacksquare$$

1.4 Displacement field

$$u = 10^{-4}\left(-x^2 + 2y^2 + 6xy\right)$$

$$v = 10^{-4}\left(3x + 6y - y^2\right)$$

$$\frac{\partial u}{\partial x} = 10^{-4}(-2x + 6y) \quad \frac{\partial u}{\partial y} = 10^{-4}(4y + 6x)$$

$$\frac{\partial v}{\partial x} = 3\times10^{-4} \qquad \frac{\partial v}{\partial y} = 10^{-4}(6 + 2y)$$

$$\varepsilon = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix}$$

at $x = 1, y = 0$

$$\varepsilon = 10^{-4}\begin{Bmatrix} -2 \\ 6 \\ 9 \end{Bmatrix}$$

1.5 On inspection, we note that the displacements u and v are given by

$$u = 0.1\,y + 4$$
$$v = 0$$

It is then easy to see that

$$\varepsilon_x = \frac{\partial u}{\partial x} = 0$$
$$\varepsilon_y = \frac{\partial v}{\partial y} = 0$$
$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0.1$$

1.6 The displacement field is given as

$$u = 1 + 3x + 4x^3 + 6xy^2$$
$$v = xy - 7x^2$$

(a) The strains are then given by

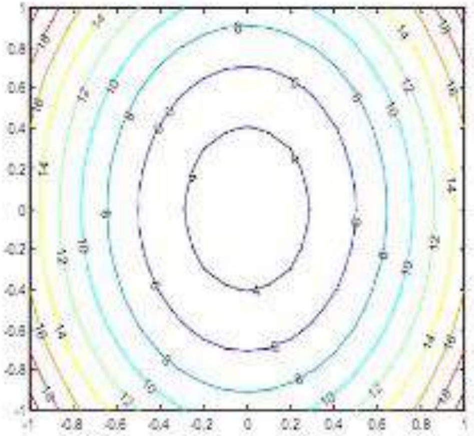
$$\varepsilon_x = \frac{\partial u}{\partial x} = 3 + 12x^2 + 6y^2$$
$$\varepsilon_y = \frac{\partial v}{\partial y} = x$$
$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 12xy + y - 14x$$

(b) In order to draw the contours of the strain field using MATLAB, we need to create a script file, which may be edited as a text file and save with ".m" extension. The file for plotting ε_x is given below

file "prob1p5b.m"

```
[X,Y] = meshgrid(-1:.1:1,-1:.1:1);
Z = 3.+12.*X.^2+6.*Y.^2;
[C,h] = contour(X,Y,Z);
clabel(C,h);
```

On running the program, the contour map is shown as follows:



Contours of ε_x

Contours of ε_y and γ_{xy} are obtained by changing Z in the script file. The numbers on the contours show the function values.

(c) The maximum value of ε_x is at any of the corners of the square region. The maximum value is 21.

■

Q6. Define Weighted - Residual method?

Ans:

Generally, the solutions obtained by solving most of the problems of engineering field are approximate solutions and it is difficult to get accurate solutions, as error exists in the solution.

Weighted residual methods are used to reduce these errors in the problems. This method consists of substituting a trial function in the differential equations formulated for the system and residual obtained is equated to zero. Thus, a solution which is very close to the exact solution, is obtained.

Q9. Why is variational formulation referred to as weak formulation?

Ans: **[Nov./Dec.-18, (R13), Q2| Model Paper-III, Q1]**

Weakening or reducing the governing differential equation of the problem by the process of integration is referred as weak formulation. In the variational formulation, differential equation of the physical problem is rewritten in the form of an equivalent integral. Thereby, upon integration, differential equation gets reduced, i.e., reduction of double differentiation into single differentiation occurs. Therefore, variational formulation referred to as weak formulation. Using variational formulation it is possible to obtain approximate solutions of spatially continuous type with less difficulty and evaluated approximate solutions forms continuous type functions of coordinates of position within the domain.

General expression for weighted residual methods,

$$\int_{\Omega} R(x) w_i(x) dx = 0 \qquad i = 1,2,\ldots,n$$

Weighting functions associated with different weighted residual techniques are,

S.No	Weighted Residual Technique	Expression	Weighting Function
1.	Point Collocation Method	$\int_{\Omega} \delta(x - x_i)\, R(x, a_i) dx = 0$	$w_i = \delta(x - x_i)$
2.	Sub-Domain Method	$\int_{\Omega} R(x, a_i) dx = 0$	$w_i = 1$
3.	Least square Method	$I = \int_{\Omega} [R(x, a_i)]^2 dx = \text{min imum}$	$w_i = 1$
4.	Galerkin's Method	$\int_{\Omega} y(x) R(x, a_i) dx = 0$	$w_i = y(x)$

Q10. Write about the concept of potential energy?

Ans: It states that, the total potential energy within the body becomes stable or minimum, when the displacement equations satisfy the equilibrium equations. These displacement equations that satisfy equations of equilibrium basically fulfills the boundary conditions and are internally compatible.

Q11. Write the potential energy for beam of span L simply supported at both ends, subjected to a concentrated load P at mid span. Assume EI as constant.

Ans: Potential energy 'π' for a beam of span '*L*', simply supported at both ends and subjected to a concentrated load '*P*' at mid span is given by,

$$\pi = U - H$$

$$\pi = \frac{P^2 L^3}{\pi^4 EI}$$

Where,

EI – Flexural rigidity of the beam (constant)

U – Strain energy

H – Work potential.

Q12. Mention the basic steps of Rayleigh-Ritz method.

Ans: The basic steps of Rayleigh-Ritz method are,

1. Assumption of a displacement field
2. Determination of the total potential
3. Solving the system of equations.

Boundary Value Problem : If a governing equation which is formulated over the domain, consists of dependent variables which compulsorily takes and its partial derivative which may probably takes particular values on the boundary of domain , then such equation describes 'boundary value problem'.

Examples of Boundary Value Problem: Analysis of axial deformation of concrete pier, study of steady-state heat flow in a bar, drawn from solid mechanics and heat transfer.

Initial Value Problem : A governing equation, formulated over a domain is said to define a initial value problem, if dependent variables are compulsorily needed and its partial derivatives are probably needed to specified initially, that is at time $t = 0$. Generally, initial value problems depends on time.

Example of Initial Value Problem: Analysis of linear motion of simple pendulum drawn from dynamics.

Boundary and Initial Value problem : If the differential equation formulated for a problem, contain dependent variables which are, needed to take specific values on the boundary and required to specified initially, then such problem is said to be both boundary and initial value problem.

Example of Boundary and Initial Value Problem: Unsteady-state heat transfer in a bar drawn from heat transfer.

Eigen Value Problem : A problem is said to be eigen value problem, if an unknown parameter exists in formulated governing equation in addition to unknown dependent variable. In eigen value problem it is needed to determine both unknown parameter and dependent variable. while satisfying differential equation and related boundary conditions.

Example of Eigen Value Problem: Analysis of axial vibrations of a bar drawn from dynamics.

Finite element method was initially developed for the analysis of aircraft structures, but the wider nature of the theory enables it to be applied for variety of boundary value problems in engineering, where the solution has to be obtained in the region or domain of a system subject to the fulfillment of certain boundary conditions. The application of finite element methods is more in the following categories of boundary value problems.

(i) Steady-state or equilibrium or time independent problems

(ii) Eigen-value problems

(iii) Transient or propagation problems

The following table gives details about the specific application of FEM in different categories.

S.No.	Field of Application	Type of Boundary Value Problem		
		Equilibrium Problems	**Eigenvalue Problems**	**Propagation Problems**
1.	Aircraft structure	In static analysis of a aircraft wings, fins, missile and rocket structures.	Natural frequencies, flutter, and in the stability of rocket, spacecraft and missile structures.	In the study of response of aircraft structures to random loads, dynamic response of aircraft and space craft to periodic loads.
2.	Mechanical design	In stress analysis of pistons, pressure vessels, gears, composite materials and linkages.	Natural frequencies and stability of gears, machine tools and linkages.	Problems of crack and fracture under dynamic loading.

weighted residual method is given by,

$$\int_{\Omega} W_i R(x, a_i)\, dx = 0 \qquad i = 1, 2, 3, \ldots, n$$

Where,

W_i – Weighting function

Ω – Domain

a_i – Unknown coefficients

weighted residual method is given by,

$$\int_{\Omega} W_i R(x, a_i)\, dx = 0 \qquad i = 1, 2, 3, \ldots, n$$

Weighting functions associated with different weighted residual techniques are,

S.No	Weighted Residual Technique	Expression	Weighting Function
1.	Point Collocation Method	$\int_{\Omega} \delta(x - x_i)\, R(x, a_i)\, dx = 0$	$w_i = \delta(x - x_i)$
2.	Sub-Domain Method	$\int_{\Omega} R(x, a_i)\, dx = 0$	$w_i = 1$
3.	Least square Method	$I = \int_{\Omega} [R(x, a_i)]^2\, dx = \text{minimum}$	$w_i = 1$
4.	Galerkin's Method	$\int_{\Omega} y(x) R(x, a_i)\, dx = 0$	$w_i = y(x)$

For the differential equation $\frac{d^2y}{d^2x} + 500x^2 = 0$ for $0 < X < 10$ and with boundary conditions $y(0) = 0$ and $y(10) = 0$, find the solution of this problem using any two weighted residual methods.

For the differential equation $\frac{d^2y}{d^2x} + 500x^2 = 0$ for $0 < X < 10$ and with boundary conditions $y(0) = 0$ and $y(10) = 0$, find the solution of this problem using any two weighted residual methods.

Given that,

Differential equation,

$$\frac{d^2y}{dx^2} + 500x^2 = 0;\ 0 \le x \le 10$$

Boundary conditions, $y(0) = 0$ and $y(10) = 0$

Consider a trial function,

$$y = a_1 x (10 - x)$$

$$y = 10a_1 x - a_1 x^2$$

Differentiating with respect to 'x',

$$\frac{dy}{dx} = 10a_1 - 2a_1 x$$

Again differentiating with respect to 'x',

$$\frac{d^2y}{dx^2} = -2a_1$$

Substituting in equation (1),

Residual, $R = -2a_1 + 500x^2$

1. Point Collocation Method

In this method residual is set to 0.

$$\text{i.e., } R = 0$$

$$-2a_1 + 500x^2 = 0$$

One collocation point is required, since there exist one unknown coefficient in the residual. Collocation point should lied between 0 and 10.

Assume, collocation point, $x = 5$

$$-2a_1 + 500(5)^2 = 0$$

$$500 \times 25 = 2a_1$$

$$\therefore \; a_1 = 6250$$

$$\therefore \text{Trial function,} \quad y = 6250\, x\,(10 - x)$$

2. Sub-domain Collocation Method

This method involves setting, integral of residual over sub-domain to zero.

$$\int_0^{10} R dx = 0$$

$$\int_0^{10} (-2a_1 + 500x^2)\, dx = 0$$

$$-2a_1 (x)_0^{10} + 500 \times \left(\frac{x^3}{3}\right)_0^{10} = 0$$

$$-2a_1(10) + \frac{500}{3}(10^3) = 0$$

$$-20a_1 + \frac{500}{3} \times 10^3 = 0$$

$$20a_1 = \frac{500 \times 10^3}{3}$$

$$a_1 = \frac{25000}{3}$$

$\therefore$ Trial function, $y = \frac{25000}{3} x\,(10 - x)$

3. Least Square Method

In this method, integral of square of weighted residual over the domain is minimum

$$I = \int_0^{10} R^2\,dx$$

$$= \int_0^{10} (-2a_1 + 500x^2)^2\,dx$$

$$= \int_0^{10} (4a_1^2 + 250000x^4 - 2000x^2 a_1)\,dx$$

$$= 4a_1^2 (x)_0^{10} + 250000\left(\frac{x^5}{5}\right)_0^{10} - 2000\left(\frac{x^3}{3}\right)_0^{10} a_1$$

$$I = 4a_1^2 \times 10 + 50000 \times 10^5 - \frac{2000}{3} \times 10^3 a_1$$

For stationary value of 'I',

$$\frac{\partial I}{\partial a_1} = 0$$

$$8a_1 \times 10 - \frac{2000}{3} \times 10^3 = 0$$

$$8a_1 = \frac{2000}{3} \times 10^2$$

$$a_1 = \frac{250}{3} \times 10^2$$

. Trial function, $y = \dfrac{25000}{3}\, x\,(10 - x)$

4. Galerkin's Method

In Galerkin's method, the domain integral which is the product of trial function and residual is set to zero.

$$\int_0^{10} y\, R dx = 0$$

$$\int_0^{10} y\,(-2a_1 + 500x^2)\, dx = 0$$

$$\int_0^{10} a_1 x(10 - x)\,(-2a_1 + 500x^2)\, dx = 0$$

$$\int_0^{10} (10a_1 x - a_1 x^2)\,(-2a_1 + 500x^2)\, dx = 0$$

$$\int_0^{10} \left(-20\, a_1^2 x + 5000 a_1 x^3 + 2a_1^2 x^2 - 500 a_1 x^4\right) dx = 0$$

$$\int_0^{10} \left(-20\, a_1^2 x + 5000 a_1 x^3 + 2a_1^2 x^2 - 500 a_1 x^4\right) dx = 0$$

$$-20a_1^2\left(\frac{x^2}{2}\right)_0^{10} + 5000a_1\left(\frac{x^4}{4}\right)_0^{10} + 2a_1^2\left(\frac{x^3}{3}\right)_0^{10} - 500a_1\left(\frac{x^5}{5}\right)_0^{10} = 0$$

$$-10a_1^2 \times 10^2 + 1250\, a_1 \times 10^4 + \frac{2}{3} a_1^2 \times 10^3 - 100a_1 \times 10^5 = 0$$

$$-\frac{1000}{3} a_1^2 + 25 \times 10^5 a_1 = 0$$

$$\therefore \quad a_1 = 7500$$

$\therefore$ Trial function, $y = 7500x\,(10 - x)$.

The following differential equation is available for a physical phenomenon.

$$\frac{d^2y}{dx^2} + 50 = 0 \leq x \leq 10$$

The Trial function is $y = a_1 x(10 - x)$

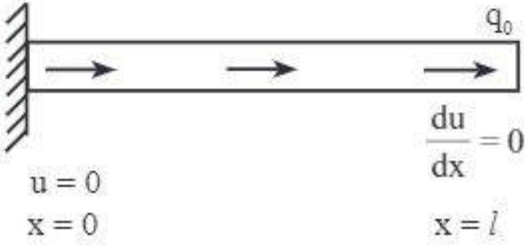
The boundary conditions are: $\begin{matrix} y(0) = 0 \\ y(10) = 0 \end{matrix}$

Find the value of the parameter a_1 by the following methods.

(i) Least square method

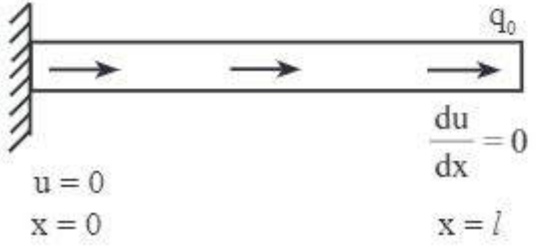
(ii) Galerkin's method.

Consider a bar subjected to a uniform axial load as shown in the figure, which can steadily show that the deformation of a body is given by differential equation $AE\frac{d^2u}{dx^2}+q_0 = 0$ with boundary conditions $u(0) = 0$, $\frac{du}{dx}=0$ at $x = l$. Find the approximate solution by using weighted residual method.



Figure

Consider a bar subjected to a uniform axial load as shown in the figure, which can steadily show that the deformation of a body is given by differential equation $AE\frac{d^2u}{dx^2}+q_0=0$ with boundary conditions $u(0)=0$, $\frac{du}{dx}=0$ at $x=l$. Find the approximate solution by using weighted residual method.



Figure

Ans:

Given that,

Differential equation,

$$AE\frac{d^2u}{dx^2}+q_0 \quad =0$$

Boundary conditions,

$$u(0)=0,\ \frac{du}{dx}(l)=0$$

Let, the trial function be,

$$u(x) = a_0 + a_1x + a_2x^2$$

Differentiating equation (2),

$$\frac{du}{dx}=a_1+2a_2x$$

Again differentiating,

$$\frac{d^2u}{dx^2} = 2a_2 \qquad ...(3)$$

Subjecting to boundary condition $u(0) = 0$,

$$0 = a_0 + a_1(0) + a_2(0)^2$$

$$a_0 = 0$$

Subjecting to boundary condition $\frac{du}{dx}(l) = 0$

$$0 = a_1 + 2a_2(l)$$

$$a_1 = -2a_2 l$$

On substituting coefficients in equation (2),

$$u(x) = 0 + (-2a_2 l)x + a_2 x^2$$

$$u(x) = a_2 x^2 - 2a_2 lx \qquad ...(4)$$

On substituting equation (3) in equation (1),

Residual,

$$R = AE(2a_2) + q_0$$

$$R = 2a_2 AE + q_0$$

Using point collocation method (a weighted residual method) in which residual is set to zero.

$$R = 0$$

$$2a_2 AE + q_0 = 0$$

$$a_2 = -\frac{q_0}{2AE}$$

On substituting 'a_2' value in equation (4), approximate solution for elongation at any distance 'x' is obtained.

$$u(x) = \left(-\frac{q_0}{2AE}\right)x^2 - 2\left(-\frac{q_0}{2AE}\right)lx$$

$$\therefore u(x) = -\frac{q_0}{2AE}(x^2 - 2lx)$$

Elongation at free end i.e., at $x = l$,

$$u_l = -\frac{q_0}{2AE}(l^2 - 2l(l))$$

$$= -\frac{q_0}{2AE}(-l^2)$$

$$\therefore u_l = \frac{q_0 l^2}{2AE}$$

Discuss the following methods to solve the given differential equation : $EI\frac{d^2y}{dx^2} - M(x) = 0$

With the boundary condition y(0) = 0 and y(l) = 0

1. Sub-domain method
2. Point collocation method.

$$EI\frac{d^2y}{dx^2} - M_{(x)} = 0$$

Discuss the following methods to solve the given differential equation : $EI\frac{d^2y}{dx^2} - M_{(x)} = 0$

With the boundary condition y(0) = 0 and y(l) = 0

1. Sub-domain method

2. Point collocation method.

Ans:

Given that,

Integral equation, $EI\frac{d^2y}{dx^2} - M(x) = 0$

Boundary conditions,

$y(0) = 0$ and $y(x) = 0$

Let, $y = a\sin\left(\frac{\pi x}{l}\right)$ be the trial function for deflection.

Differentiating 'y',

$$\frac{dy}{dx} = a\frac{\pi}{l}\cos\left(\frac{\pi x}{l}\right)$$

$$\frac{d^2y}{dx^2} = -a\frac{\pi^2}{l^2}\sin\left(\frac{\pi x}{l}\right)$$

Therefore,

$$\text{Residual, } R = EI\frac{d^2y}{dx^2} - M(x)$$

$$R = EI\left[-\frac{\pi^2 a}{l^2}\sin\left(\frac{\pi x}{l}\right)\right] - M(x) \quad \text{... (1)}$$

1. Sub-domain Method

In this method, integral of residual over domain is set to zero.

$$\int_0^l R\,dx = 0$$

$$\int_0^l \left[-\frac{\pi^2 a}{l^2} EI \sin\left(\frac{\pi x}{l}\right) - M(x)\right] dx = 0$$

$$-\frac{\pi^2 a}{l^2} EI \left[-\cos\left(\frac{\pi x}{l}\right)\left(\frac{l}{\pi}\right)\right]_o^l - [Mx]_o^l = 0$$

$$\frac{\pi a EI}{l}\left[\cos(\pi) - \cos(0)\right] - Ml = 0$$

$$\frac{\pi a EI}{l}(-1-1) - Ml = 0$$

$$-\frac{2\pi a EI}{l} = Ml$$

$$a = -\frac{Ml^2}{2\pi EI} = -0.159\,\frac{Ml^2}{EI}$$

Trial function, $y = -0.159\frac{Ml^2}{EI}\sin\left(\frac{\pi x}{l}\right)$

1. Sub-domain Method

In this method, integral of residual over domain is set to zero.

$$\int_0^l R\,dx = 0$$

$$\int_0^l \left[-\frac{\pi^2 a}{l^2} EI \sin\left(\frac{\pi x}{l}\right) - M(x)\right] dx = 0$$

$$-\frac{\pi^2 a}{l^2} EI \left[-\cos\left(\frac{\pi x}{l}\right)\left(\frac{l}{\pi}\right)\right]_o^l - [Mx]_o^l = 0$$

$$\frac{\pi a EI}{l}\left[\cos(\pi) - \cos(0)\right] - Ml = 0$$

$$\frac{\pi a EI}{l}(-1-1) - Ml = 0$$

$$-\frac{2\pi a EI}{l} = Ml$$

$$a = -\frac{Ml^2}{2\pi EI} = -0.159\,\frac{Ml^2}{EI}$$

Trial function, $y = -0.159\frac{Ml^2}{EI}\sin\left(\frac{\pi x}{l}\right)$

Trial function, $y = -0.159\frac{Ml^2}{EI}\sin\left(\frac{\pi x}{l}\right)$

At $x = l/2$, $y = y_{max}$

i.e., $y_{max} = -0.159\frac{Ml^2}{EI}\sin\left(\frac{\pi l}{2l}\right)$

$\therefore \quad y_{max} = -0.159\,\frac{Ml^2}{EI} \qquad \left(\because \sin\frac{\pi}{2} = 1\right)$

2. Point Collocation Method

In this method, the residual is set to zero.

$$R = 0$$

$$-\frac{\pi^2 a}{l^2} EI \sin\left(\frac{\pi x}{l}\right) - M(x) = 0$$

$$-\frac{\pi^2 a}{l^2} EI \sin\left(\frac{\pi x}{l}\right) = M$$

At $x = l/2$, $y = y_{max}$. Therefore, substitute $x = l/2$ in the above equation.

$$-\frac{\pi^2 a}{l^2} EI \sin\left(\frac{\pi l}{2l}\right) = M$$

$$-\frac{\pi^2 aEI}{l^2} = M$$

$$\therefore \quad a = -\frac{Ml^2}{\pi^2 EI}$$

Trial function, $y = -\frac{Ml^2}{\pi^2 EI} \sin\left(\frac{\pi x}{l}\right)$

2. Point Collocation Method

In this method, the residual is set to zero.

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At $x = l/2$, $y = y_{\max}$. Therefore, substitute $x = l/2$ in the above equation.

$$-\frac{\pi^2 a}{l^2} EI \sin\left(\frac{\pi l}{2l}\right) = M$$

$$-\frac{\pi^2 aEI}{l^2} = M$$

$$\therefore \quad a = -\frac{Ml^2}{\pi^2 EI}$$

Trial function, $y = -\frac{Ml^2}{\pi^2 EI} \sin\left(\frac{\pi x}{l}\right)$

Trial function, $y = -\frac{Ml^2}{\pi^2 EI} \sin\left(\frac{\pi x}{l}\right)$

At $x = l/2$, $y = y_{\max}$

$$y_{\max} = \frac{Ml^2}{\pi^2 EI} \sin\left(\frac{\pi l}{2l}\right)$$

$$\therefore \quad y_{\max} = -0.101 \frac{Ml^2}{EI} \qquad \left(\because \sin\frac{\pi}{2} = 1\right)$$

Variational Methods

Variational method involves rewriting the differential equation of physical problem in the form of equivalent integral.

Obtained integral is termed as **functional** and is allowed to become stationary. Functional becomes stationary at extremum conditions i.e., minimum or maximum conditions.

Therefore, functional is allowed to reach extremum conditions by using appropriate trial functions.

For a problem, trial function which is employed to make the integral stationary is termed as approximate solution.

Any approximation method which uses the principle such as principle of minimum potential energy and principle of virtual work are said to be variational principles.

Rayleigh-Ritz method

is a variational method and is employed to evaluate solution of structural problems. Potential energy will be stored in all structures when acted upon by the load. Stored potential energy is considered as functional. An approximate solution or trial function is assumed so that functional becomes minimum when trial function is substituted in it.

Usage of variational method is limited to the problems governed by the differential equations with order greater than one. If there exists first derivative in the differential equation, it is not possible to apply variational method for that problem.

Principle of Minimum Potential Energy

For conservative systems, of all the kinematically admissible displacement fields, those corresponding to equilibrium extremize the total potential energy. If the extremum condition is a minimum, the equilibrium state is stable.

Kinematically admissible displacements are those that satisfy the single-valued nature of displacements (compatibility) and the boundary conditions.

Principle of Virtual Work

A body is in equilibrium if the internal virtual work equals the external virtual work for every kinematically admissible displacement field $\langle \boldsymbol{\phi}, \boldsymbol{\epsilon}(\phi) \rangle$.

Principle of virtual work:

For every displacement field of kinematically admissible type. A body is said to be in equilibrium, if value of the internal virtual work becomes same as that of external virtual work.

$$\therefore \ \partial U = \partial W$$

The internal virtual strain energy,

$$\partial U = \int_V (\partial \varepsilon)^T \sigma \, dV \qquad .. (1)$$

Where,

U – Strain energy (internal work)

W – External work

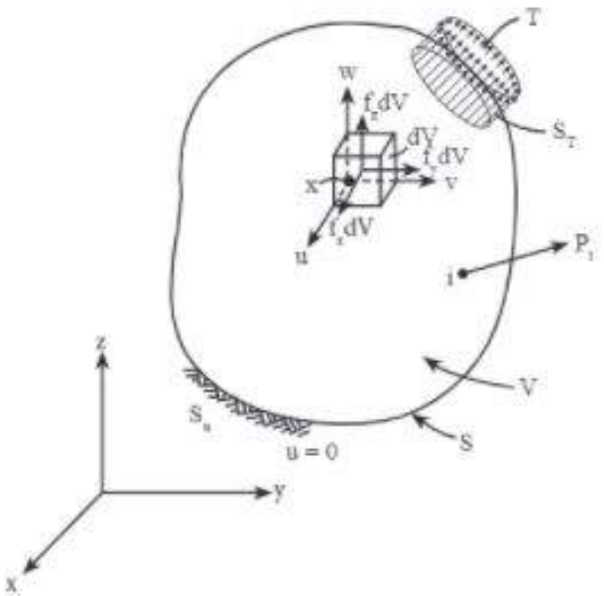
ε – Strain vector

σ – Stress vector

V – Volume of Element.

Total external virtual workdone,

$$\partial W = (\partial u)^T P_i + \int_V (\partial u)^T b dV + \int_A (\partial u)^T q dA \quad \text{... (4)}$$



From minimum potential energy principle,

Total potential energy, $\pi = U - W_p$ ... (5)

Where,

U – Strain energy of the system

W_p – Total work potential of the load.

Total potential energy for the general elastic body is given by,

$$\pi = \frac{1}{2}\int_V \sigma^T \varepsilon dV - \int_V u^T b dV - \int_A u^T q dA - \sum_i u_i^T P_i \quad \text{... (6)}$$

i.e., $\frac{\partial \pi}{\partial x} = 0$

Where,

π – Total potential energy

x – Displacement field

Also, $\pi = U - W$

Where,

U – Strain energy within the body

W – Work done on the system

Derivation of Equilibrium Equation

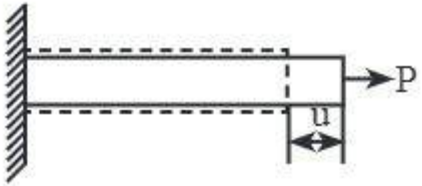
For the body shown in figure,

Let,

P – Load acting on the body

u – Deformation

K – Stiffness



Figure

Workdone on the body,

$$W = \text{Applied load} \times \text{Deformation length}$$

$$= P \times u$$

Strain energy within the body,

$$U = \frac{1}{2}\,[\text{Force in the body} \times \text{deformation length}]$$

$$= \frac{1}{2}\,[Ku \times u]$$

$$= \frac{1}{2}\,Ku^2$$

But, total potential energy, $\pi = U - W$

$$= \frac{1}{2}\,Ku^2 - P.u$$

Under equilibrium condition, for potential energy to be minimum,

$$\frac{\partial \pi}{\partial u} = 0$$

$$\frac{\partial}{\partial u}\left[\frac{1}{2}Ku^2 - P.u\right] = 0$$

$$Ku - P = 0$$

$$\therefore\ Ku = P$$

$\therefore$ Equilibrium equation is given by, $Ku = P$.

Finite Element Discretization Corresponding to Variational Formulation

The mathematical model of a bar is discretized and assembled to form a model, which comprises of small bar elements. Then, equations for finite elements of the bar is derived by using total potential energy functional.

Figure (2): Bar Member

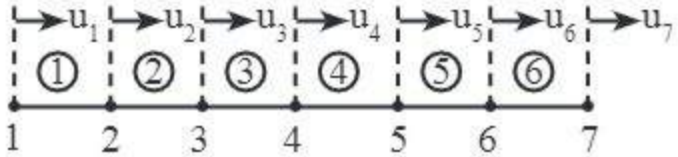


Figure (3): Discretized Bar Member

Figure (2): Bar Member

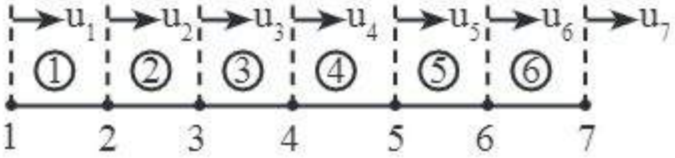


Figure (3): Discretized Bar Member

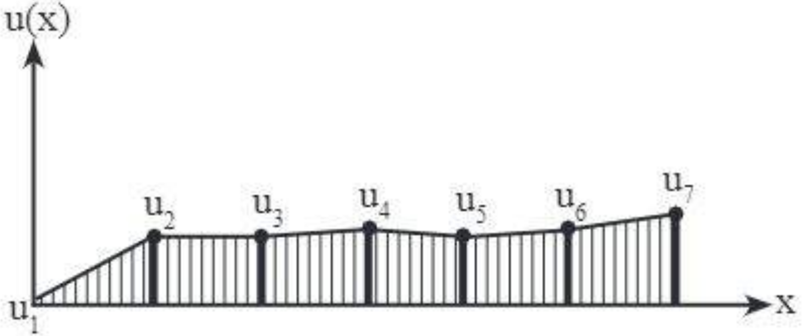


Figure (4): Representation of u (x), the Displacement Trial Function

TPE-total potential energy

Consider a bar member, divided into certain number of elements, as shown in figure. TPE functionals are scalar quantities and for a discretized bar member, TPE functional is the summation of functional of each element.

i.e., $\pi = \pi_1 + \pi_2 + ... + \pi_{n-1} + \pi_n$

Where,

n – Total number of elements

i.e., $\pi = \pi_1 + \pi_2 + ... + \pi_{n-1} + \pi_n$

Similarly, internal energy, external energy and condition of minimum potential energy principle are formed by summation of the corresponding parameter of each finite element.

Minimum potential energy principle is given by,

$$\delta\pi = \delta\pi_1 + \delta\pi_2 + ... + \delta\pi_{n-1} + \delta\pi_n = 0$$

For an element '*e*' as a whole, based on variational calculus fundamentals, the above equation can be written as

$$\delta\pi_e = \delta U_e - \delta W_e = 0$$

This equation is called variational equation and it is a basic formulation, from which stiffness equations for the elements can be developed, after discretization of displacement field for the bar member.

Rayleigh – Ritz method is a variational method and mostly used to solve structural problems which are complex in nature.

There are two techniques of problem solving in Rayleigh-Ritz method.

1. **Rayleigh-Ritz Method Using Minimum Potential Energy Concept**

2. **Rayleigh-Ritz Method Using Integral Approach**

1. Rayleigh-Ritz Method Using Minimum Potential Energy Concept

In this method, initially a trial function in terms of Ritz parameters (coefficients) is considered. Structure of assumed trial function may be a polynomial function or trigonometric function.

Structure of polynomial trial function,

$$y = a_1 + a_2 x + a_3 x^2 + a_4 x^3 + \ldots \qquad \ldots(1)$$

Structure of trigonometric trial function,

$$y = a_1 \sin\frac{\pi x}{l} + a_2 \sin\frac{3\pi x}{l} + a_3 \sin\frac{5\pi x}{l} + \ldots \qquad \ldots(2)$$

Where,

$a_1, a_2, a_3 \ldots a_n$ – Ritz parameters or coefficients

Then, total potential energy is formulated using anyone of two trial function structures mentioned above.

Total potential energy,

$$\pi = U - W$$

Where, 'U' and 'W' are strain energy and workdone due to external force respectively and are specified so that they forms the functions of approximated trial function. Then formulation of total potential energy is carried out by deriving the trial function compatible with the formats of 'U' and 'W'. Finally, approximate solution can be evaluated when total potential energy is made to reach minimum value.

For potential energy to be minimum,

$$\frac{\partial \pi}{\partial a_1} = \frac{\partial \pi}{\partial a_2} = \ldots. = \frac{\partial \pi}{\partial a_n} = 0$$

Thus, from above equation, Ritz coefficients $a_1, a_2 \ldots a_n$ can be calculated.

2. Rayleigh-Ritz Method Using Integral Approach

This method involves rewriting the differential equation of physical problem in the form of equivalent integral. Obtained integral is termed as functional and is allowed to become stationary. Functional becomes stationary at extremum conditions i.e, minimum or maximum conditions. Therefore, functional is allowed to reach extremum conditions by using appropriate trial functions. For a problem, trial function which is employed to make the integral stationary is termed as approximate solution.

Consider a physical problem whose governing differential equation is given by,

$$K\frac{d^2y}{dx^2} + L = 0$$

Where,

K, L – Constants or variables

Subjected to boundary conditions,

$$y(0) = y_0,$$
$$y(l) = y_l$$

An integral equivalent to differential equation is given by,

$$I = \int_0^l \left[\frac{1}{2}K\left(\frac{dy}{dx}\right)^2 - Ly\right]dx$$

Where,

I – Functional

Then anyone of the types of trial functions in equations (1) and (2) is considered and differentiated so that it is suitable for equivalent integral format. Evaluation of the approximate solution is obtained by making the integral to become stationary.

$$\frac{\partial I}{\partial a_i} = 0,$$

Where, $i = 1, 2, 3 \ldots n$,

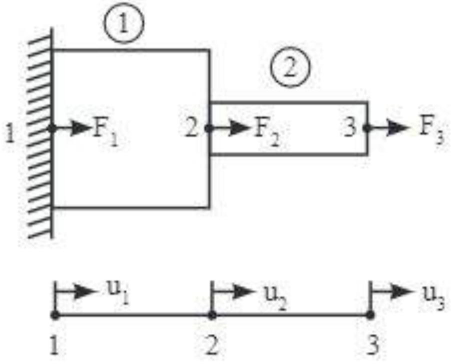
2. Rayleigh-Ritz Method Using Integral Approach

General Steps:

1. Formulate Potential Energy Functional
2. Assume a trial displacement function, which should satisfy boundary condition
3. Substitute admissible trial displacement function into Potential Energy Functional and simplify it
4. Minimize the Potential Energy Functional so as to obtain the equilibrium condition
5. Determine the unknown displacement, hence strain and stress

Explain the potential energy formulation for obtaining element equations in Finite element methods.

Consider a stepped bar with three nodes and two elements.



Figure

F_1, F_2, F_3 – Concentrated loads at each node

u_1, u_2, u_3 – Displacements at each node

Total potential energy = Strain energy + Workdone

$$\pi = U - W$$

('–ve' sign due to workdone on the system)

For minimum potential energy,

$$\frac{\partial \pi}{\partial s} = 0$$

Since, the bar is divided into two elements,

Total potential energy, $\pi = \pi_1 + \pi_2$ ... (1)

Considering element (1),

Element potential energy,

$$\pi_1 = U_1 - W_1$$
$$= \left[\frac{1}{2}k_1\left(u_2 - u_1\right)^2 - \left(F_1u_1 + F_2u_2\right)\right]$$
$$\pi_1 = \frac{1}{2}k_1\left(u_2 - u_1\right)^2 - F_1u_1 - F_2u_2$$

For potential energy to be minimum at each node of element (1),

i.e., at node 1,

$$\frac{\partial \pi_1}{\partial u_1} = 0$$

$$\frac{\partial}{\partial u_1}\left[\frac{1}{2}k_1(u_2 - u_1)^2 - F_1u_1 - F_2u_2\right] = 0$$

$$k_1(u_2 - u_1)(-1) - F_1 = 0$$

$$k_1u_1 - k_1u_2 = F_1 \quad \text{... (2)}$$

And, at node 2,

$$\frac{\partial \pi_1}{\partial u_2} = 0$$

$$\frac{\partial}{\partial u_2}\left[\frac{1}{2}k_1(u_2 - u_1)^2 - F_1u_1 - F_2u_2\right] = 0$$

$$k_1(u_2 - u_1)(1) - F_2 = 0$$

$$-k_1u_1 + k_1u_2 = F_2 \quad \text{... (3)}$$

k_1 – Element stiffness

$$= \frac{A_1 E_1}{l_1}$$

Finite element matrix is obtained by writing equations (2) and (3) in matrix form.

$$\begin{array}{cc} 1 & 2 \end{array}$$
$$\begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{matrix} 1 \\ 2 \end{matrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \quad \ldots (4)$$

Considering element (2),

Element potential energy,

$$\pi_2 = U_2 - W_2$$

$$= \left[\frac{1}{2}k_2(u_3 - u_2)^2 - (F_2 u_2 + F_3 u_3)\right]$$

$$\pi_2 = \frac{1}{2}k_2(u_3 - u_2)^2 - F_2 u_2 - F_3 u_3$$

u3

For potential energy to be minimum at each node of element (2),

i.e., at node 2,

$$\frac{\partial \pi_2}{\partial u_2} = 0$$

$$\frac{\partial}{\partial u_2}\left[\frac{1}{2}k_2(u_3 - u_2)^2 - F_2 u_2 - F_3 u_3\right] = 0$$

$$k_2(u_3 - u_2)(-1) - F_2 = 0$$

$$k_2 u_2 - k_2 u_3 = F_2 \qquad \dots (5)$$

And, at node 3,

$$\frac{\partial \pi_2}{\partial u_3} = 0$$

$$\frac{\partial}{\partial u_3}\left[\frac{1}{2}k_2(u_3 - u_2)^2 - F_2 u_2 - F_3 u_3\right] = 0$$

$$k_2(u_3 - u_2)(1) - F_3 = 0$$

$$-k_2 u_2 + k_2 u_3 = F_3 \qquad \dots (6)$$

Finite element matrix is obtained by writing equations (5) and (6) in matrix form.

$$\begin{array}{cc} & \begin{array}{cc} 2 & 3 \end{array} \\ & \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{array}{c} 2 \\ 3 \end{array} \end{array} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix} \qquad \ldots (7)$$

Global finite element matrix is given by,

$$[K]\ \{u\} = \{F\}$$

It obtained by adding equations (4) and (7).

$$\begin{array}{cc} \begin{matrix} 1 & \quad 2 \end{matrix} & \\ \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} & \begin{matrix} 1 \\ 2 \end{matrix} \end{array} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \qquad \dots (4)$$

$$\begin{array}{cc} \begin{matrix} 2 & \quad 3 \end{matrix} & \\ \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} & \begin{matrix} 2 \\ 3 \end{matrix} \end{array} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix} \qquad \dots (7)$$

$$\begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix}$$

$$\begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}$$

Global finite element matrix is given by,

$$[K]\ \{u\} = \{F\}$$

It obtained by adding equations (4) and (7).

$$\begin{array}{cc} \begin{matrix} 1 & \qquad 2 & \qquad 3 \end{matrix} & \\ \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} & \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \end{array} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

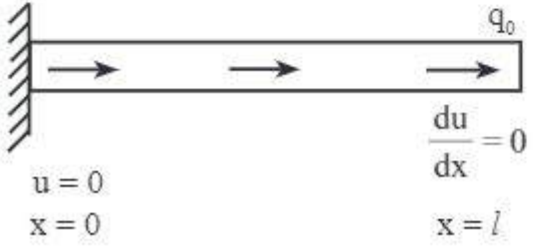
Where,

$[K]$ – Global stiffness matrix

$\{u\}$ – Global displacement vector

$\{F\}$ – Load vector

Q. Consider a bar subjected to a uniform axial load as shown in the figure, which can steadily show that the deformation of a body is given by differential equation $AE\frac{d^2u}{dx^2}+q_0 = 0$ with boundary conditions $u(0) = 0$, $\frac{du}{dx} = 0$ at $x = l$. Find the approximate solution by using weighted residual method.



Figure

Q. The functional form of a bar clamped at one end and left free at the other end and subjected to uniform axial load q is given by,

$$I = \int_0^l \left[\frac{1}{2} AE \left(\frac{du}{dx} \right)^2 - qu \right] dx$$

The essential boundary is u(0) = 0, obtain the approximate solution to the problem by using Rayleigh-Ritz method.

(Or)

Example 3.1. *A bar under uniform load.* Consider a bar clamped at one end and left free at the other end and subjected to a uniform axial load q_0 as shown in Figure 3.4. The governing differential equation is given by

$$AE\frac{d^2u}{dx^2} + q = 0$$

with the boundary conditions $u(0) = 0$; $\left.\frac{du}{dx}\right|_{x=L} = 0$.

illustrate the solution using the R–R method.

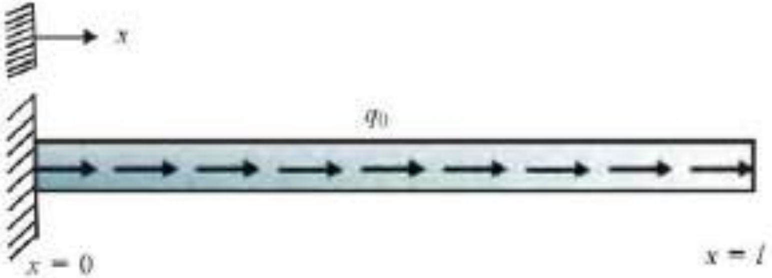


Fig. 3.4 Rod under axial load (Example 3.1).

General Steps:

1. Formulate Potential Energy Functional
2. Assume a trial displacement function, which should satisfy boundary condition
3. Substitute admissible trial displacement function into Potential Energy Functional and simplify it
4. Minimize the Potential Energy Functional so as to obtain the equilibrium condition
5. Determine the unknown displacement, hence strain and stress

Strain energy stored in the bar, $U = \int_0^L \left[\frac{1}{2} AE \left(\frac{du}{dx}\right)^2\right] dx$

Work potential of the external forces, W $= -\int_0^L q_0 u \, dx$

I (Functional) or $\Pi_P = \int_0^L \left[\frac{1}{2} AE \left(\frac{du}{dx}\right)^2 - q_0 u\right] dx$

$$u(x) \approx c_1 x + c_2 x^2$$

This satisfies the essential boundary condition that $u(0) = 0$. We have

$$\frac{du}{dx} = c_1 + 2c_2 x$$

$$\Pi_p = \int_0^L \left[\frac{AE}{2}(c_1 + 2c_2 x)^2 - q_0(c_1 x + c_2 x^2)\right] dx$$

$$= \frac{AE}{2}\left[c_1^2 L + \frac{4c_2^2 L^3}{3} + 2c_1 c_2 L^2\right] - q_0 \frac{c_1 L^2}{2} - q_0 c_2 \frac{L^3}{3}$$

$$\frac{\partial \Pi_p}{\partial c_i} = 0, \qquad i = 1, 2$$

Therefore,

$$\frac{\partial \Pi_p}{\partial c_i} = 0 \Rightarrow \frac{AE}{2}(2c_1L + 2c_2L^2) - \frac{q_0L^2}{2} = 0$$

$$\frac{\partial \Pi_p}{\partial c_2} = 0 \Rightarrow \frac{AE}{2}(8c_2L^3/3 + 2c_1L^2) - \frac{q_0L^3}{3} = 0$$

Solving, we obtain

$$c_1 = \frac{q_0L}{AE}, \qquad c_2 = -\frac{q_0}{2AE}$$

Thus,

$$u(x) = \frac{q_0}{AE}x(L - x/2) = \frac{q_0}{2AE}(2Lx - x^2)$$

Bars

$$SE = \frac{1}{2} P \delta \quad \left(\delta = \frac{Pl}{AE}\right)$$

$$= \frac{1}{2} P \cdot \frac{Pl}{AE}$$

$$= \frac{1}{2} \frac{P^2 l}{AE} \times \frac{A}{A}$$

$$= \frac{1}{2} \left(\frac{P}{A}\right)^2 \frac{lA}{E}$$

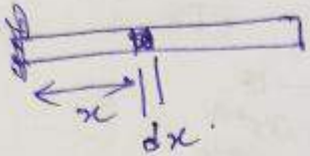
$$= \frac{1}{2} \sigma^2 \cdot \frac{V}{E} \quad \left(\because E = \frac{\sigma}{\epsilon}\right)$$

$$= \frac{1}{2} \sigma^2 \cdot \frac{V}{(\sigma/\epsilon)}$$

$$= \frac{1}{2} \sigma \cdot \epsilon \cdot V$$

$$d(SE) = \frac{1}{2} \sigma \epsilon \, dv$$

$$dv = A \cdot dx$$



$$d(SE) = \frac{1}{2} \sigma \cdot \epsilon \cdot A \cdot dx$$

$$= \frac{1}{2} \cdot E\epsilon \cdot \epsilon \cdot A \, dx$$

$$= \frac{AE}{2} \epsilon^2 dx$$

$$= \frac{AE}{2} \left(\frac{du}{dx}\right)^2 dx$$

$$\boxed{SE = \int d(SE) = \int \frac{AE}{2} \left(\frac{dy}{dx}\right)^2 dx}$$

Beams

$$SE = \frac{1}{2}\sigma \cdot \epsilon \cdot V = \frac{\sigma^2 V}{2E}$$

$$\left(\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R}\right)$$

$$\sigma = \frac{My}{I}.$$

$$SE = \frac{M^2 y^2}{I^2} \cdot \frac{V}{2E}$$

$$d(SE) = \int \frac{M^2 y^2}{2EI^2} A \cdot dx$$

$$SE = \int d(SE)$$

$$= \int \frac{M^2 I}{2EI^2}\, dx$$

$$\left(\because \int y^2\, dA = I.\right)$$

$$SE = \int \frac{M^2}{2EI}\, dx.$$

$$\boxed{EI \frac{d^2y}{dx^2} = M} \Rightarrow \frac{M}{EI} = \frac{d^2y}{dx^2}$$

$$SE = \frac{EI}{2} \int \left(\frac{M}{EI}\right)^2 dx$$

$$\boxed{SE = \frac{EI}{2} \int \left(\frac{d^2y}{dx^2}\right)^2 dx}$$

Example 3.2. *A simply supported beam under uniform load.* Consider a simply supported beam under uniformly distributed load q_0 as shown in Figure 3.5. For a deformation $v(x)$, we have

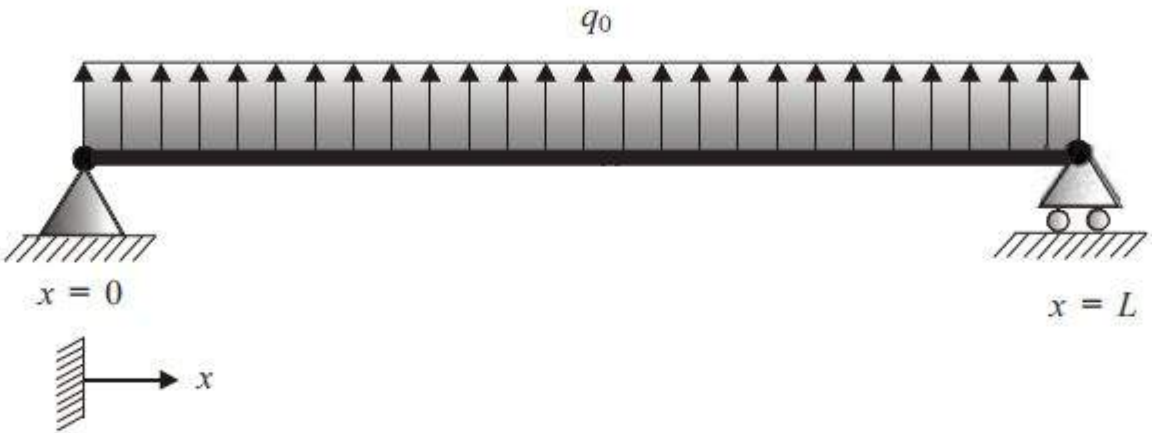


Fig. 3.5 Simply supported beam under load (Example 3.2).

The strain energy

$$U = \int_0^L \frac{1}{2} EI \left(\frac{d^2 v}{dx^2} \right)^2 dx$$

The potential of the external forces is

$$V = -\int_0^L q_0 v \, dx$$

Thus we have the total potential

$$\Pi_p = \int_0^L \left[\frac{EI}{2} \left(\frac{d^2 v}{dx^2} \right)^2 - q_0 v \right] dx$$

Assume a displacement field. Let us assume $v(x) \approx c_1 \sin(\pi x/L)$. This satisfi boundary conditions $v(0) = 0 = v(L)$. We have

$$\frac{d^2 v}{dx^2} = -c_1 \left(\frac{\pi}{L} \right)^2 \sin \frac{\pi x}{L}$$

Evaluation of the total potential. The total potential of the system is

$$\Pi_p = \int_0^L \left[\frac{EI}{2} \left(-c_1 \left(\frac{\pi}{L} \right)^2 \sin \frac{\pi x}{L} \right)^2 dx - q_0 c_1 \sin \frac{\pi x}{L} \right] dx$$

$$= \frac{\pi^4 EI}{4 L^3} c_1^2 - \frac{2 q_0 L}{\pi} c_1$$

$$\frac{\partial \Pi_p}{\partial c_1} = 0$$

$$\frac{\pi^4 EI}{2L^3} c_1 - \frac{2 q_0 L}{\pi} = 0$$

$$c_1 = 0.01307 \frac{q_0 L^4}{EI}$$

Thus the final solution is

$$v(x) = 0.01307 \frac{q_0 L^4}{EI} \sin \frac{\pi x}{L}$$

Example 1.2

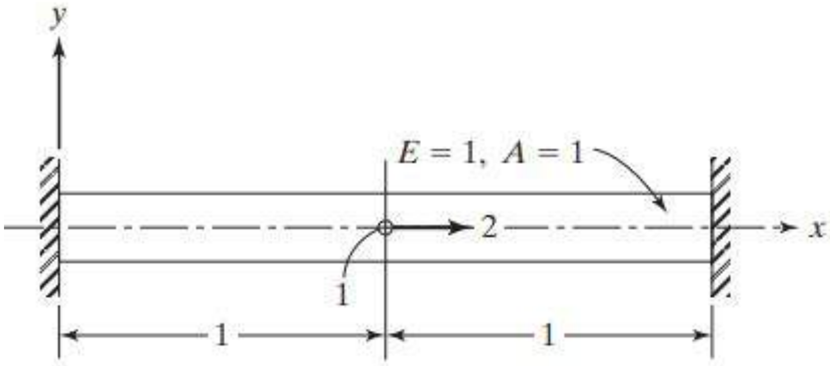
The potential energy for the linear elastic one-dimensional rod (Fig. E1.2), with body force neglected, is

$$\Pi = \frac{1}{2}\int_0^L EA\left(\frac{du}{dx}\right)^2 dx - 2u_1$$

where $u_1 = u(x = 1)$.

$u = 0$ at $x = 0$ and $u = 0$ at $x = 2$.

Write the expression for the displacement and stress?



Let us consider a polynomial function

$$u = a_1 + a_2x + a_3x^2$$

This must satisfy $u = 0$ at $x = 0$ and $u = 0$ at $x = 2$. Thus,

$$0 = a_1$$
$$0 = a_1 + 2a_2 + 4a_3$$

Hence,

$$a_2 = -2a_3$$
$$u = a_3(-2x + x^2)$$

at x=1, u=u_1

$$u_1 = -a_3$$

Then $du/dx = 2a_3(-1 + x)$ and

$$\Pi = \frac{1}{2}\int_0^2 4a_3^2(-1 + x)^2dx - 2(-a_3)$$

$$= 2a_3^2\int_0^2 (1 - 2x + x^2)dx + 2a_3$$

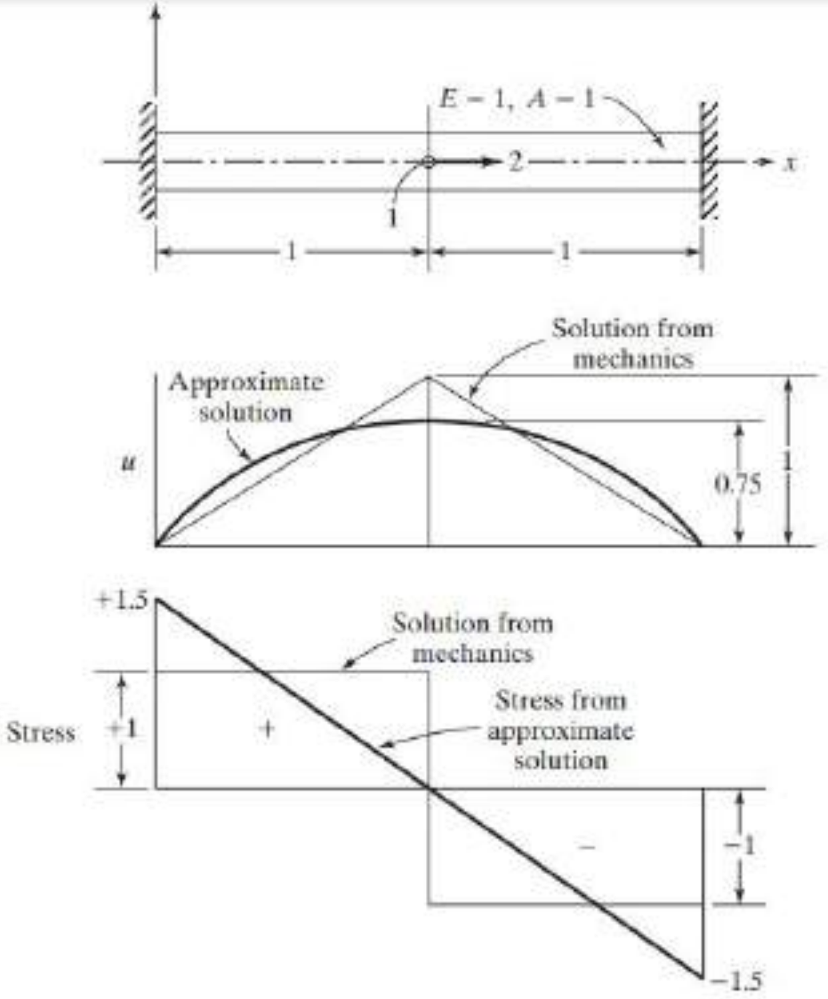
$$= 2a_3^2(\tfrac{2}{3}) + 2a_3$$

We set $\partial\Pi/\partial a_3 = 4a_3(\frac{2}{3}) + 2 = 0$, resulting in

$$a_3 = -0.75 \quad u_1 = -a_3 = 0.75$$

The stress in the bar is given by

$$\sigma = E\frac{du}{dx} = 1.5(1 - x)$$



E = 1, A = 1
2
x
1
1
1
Solution from mechanics
Approximate solution
u
0.75
1
+1.5
Solution from mechanics
Stress from approximate solution
Stress
+1
+
−
−1
−1.5

Approximative Methods

Variational Methods

approximation is based on the minimization of a functional, as those defined in the earlier slides.

- **Rayleigh-Ritz Method**

Weighted Residual Methods

start with an estimate of the the solution and demand that its weighted average error is minimized

- **The Galerkin Method**
- **The Least Square Method**
- **The Collocation Method**
- **The Subdomain Method**

1 Dimensional Problems

Ex: Bars

Trusses

Beams

Steps in an FE Analysis

Preprocessor/Modeling

Analysis run / Solve

Post-processing/View results

Preprocessor / Modeling:

- Identification of the appropriateness of analysis by FEM
- Identification of type of analysis
- Idealization, ie., choice of element type/types

Preprocessor / Modeling:

- Identification of the appropriateness of analysis by FEM
- Identification of type of analysis
- Idealization, ie., choice of element type/types
- Creation of material behavior model
- Discretization of the solution region (meshing)
- Application of boundary conditions

Analysis run / Solve:

- Formulation of element stiffness matrices
- Assembly of global stiffness matrix [K]
- formulation of load vector {F}
- Solution of [K]{u}={F} to get nodal displacements {u}

Analysis run / Solve:

- Formulation of element stiffness matrices
- Assembly of global stiffness matrix [K]
- formulation of load vector {F}
- Solution of [K]{u}={F} to get nodal displacements {u}
- (Solution of $[K] - \lambda[M]$ in case of dynamic analysis)
- (Solution of $[K] - \lambda[K_g]$ in case of buckling analysis)
- Calculation of elemental stresses

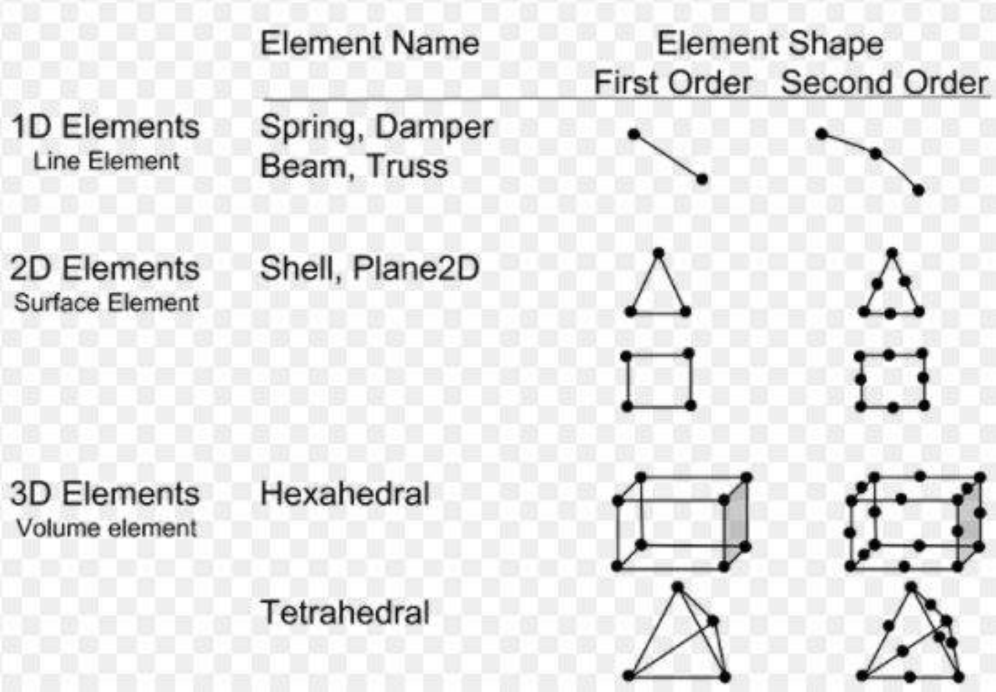
Postprocessing / View results:

- View results (displacements, stresses, mode shapes, etc.)
- Interpret and validate results
- If required, re-formulate, and re-analysis

Basic Steps Involved In FEM:

1. Domain Discretization
2. Selection of displacement functions (interpolation)
3. Formation of elemental (stiffness matrix and load vector)
4. Formation of Global (stiffness matrix and load vector) : K U = F
5. Application of boundary condition
6. Solution of simultaneous equations (for unknown nodal displacements)
7. Calculation of stresses and strains
8. Interpretation of results

1. **Domain Discretization:** It is performed by using the mesh generating programs (preprocessors). This step involves splitting the structure into number of small regular shaped elements. Generally, a body is discretized by using tetrahedron or hexahedron elements in 3D analysis, whereas, by employing triangular or quadrilateral elements in 2D analysis.



2. **Selection of displacement functions**

(Specifying the interpolation function order i.e, Linear or Quadratic approximation)

The loading consists of three types: the **body force** f, the **traction force** T, and the **point load** P_i. These forces are shown acting on a body in Fig. 3.1. A body force is a distributed

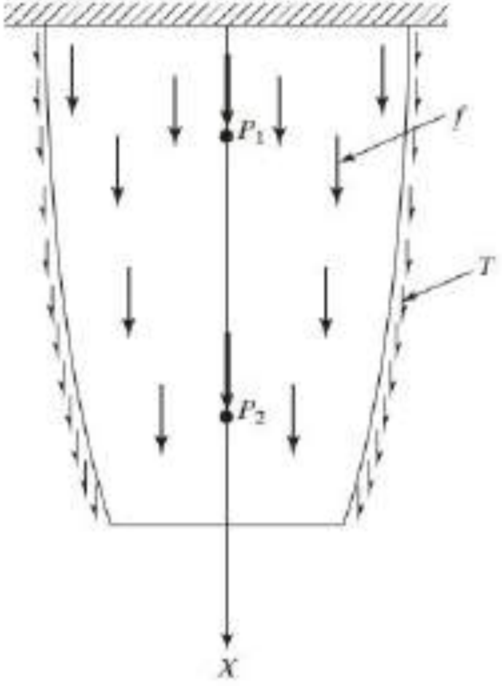
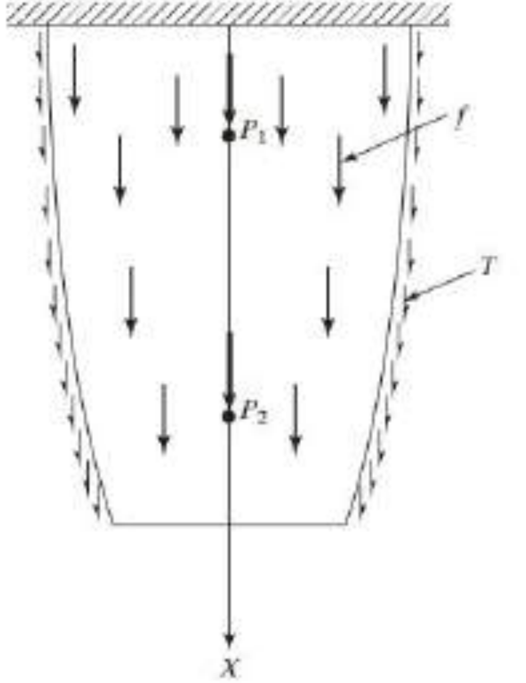


FIGURE 3.1 One-dimensional bar loaded by traction, body, and point loads.



One-dimensional bar loaded by traction, body, and point loads.

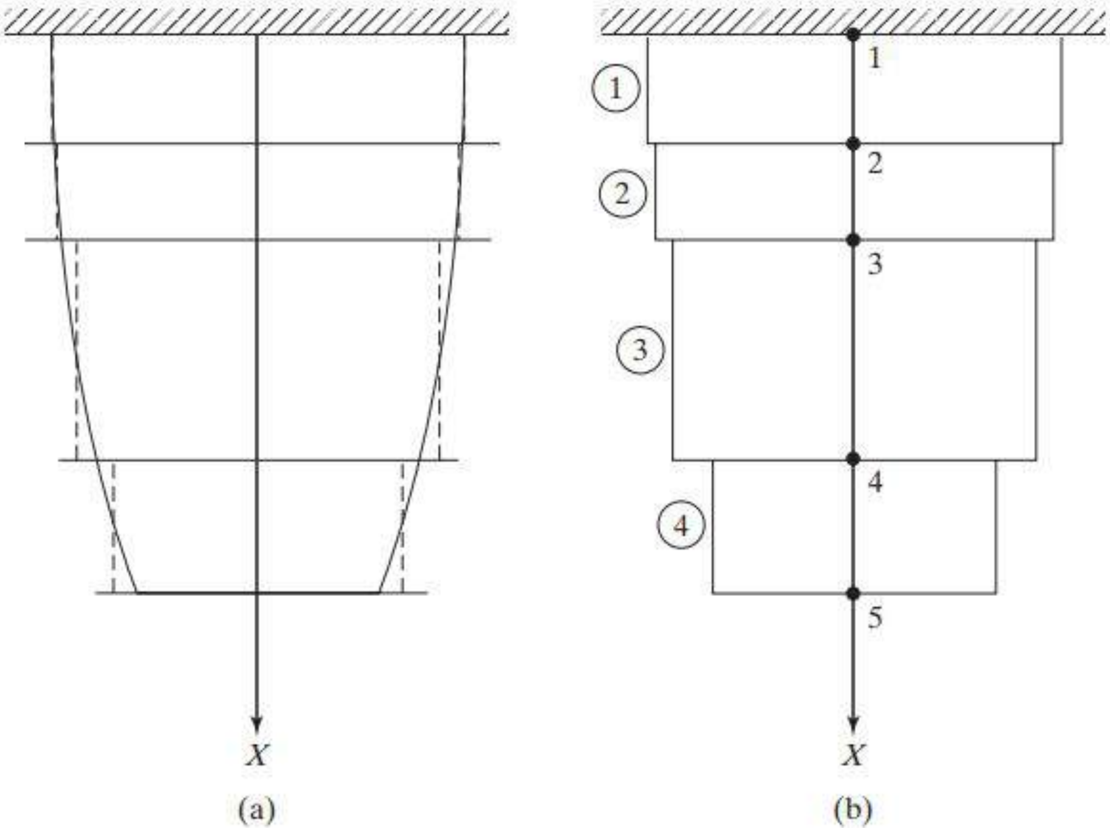


FIGURE 3.2 Finite element modeling of a bar.

element model in Fig. 3.2b has five dof. The displacements along each dof are denoted by $Q_1, Q_2, \ldots, Q_5$. In fact, the column vector $\mathbf{Q} = [Q_1, Q_2, \ldots, Q_5]^T$ is called the *global displacement vector*. The *global load vector* is denoted by $\mathbf{F} = [F_1, F_2, \ldots, F_5]^T$. The vectors $\mathbf{Q}$ and $\mathbf{F}$ are shown in Fig. 3.3 . The sign convention used is that a displacement or

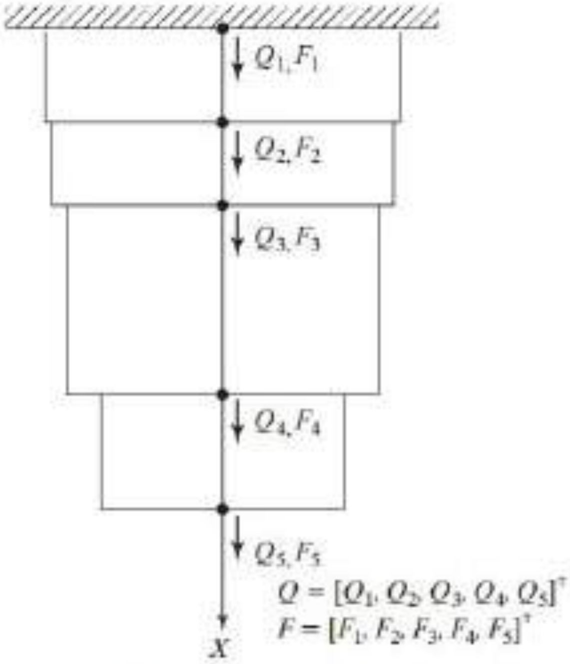


FIGURE 3.3 **Q** and **F** vectors.

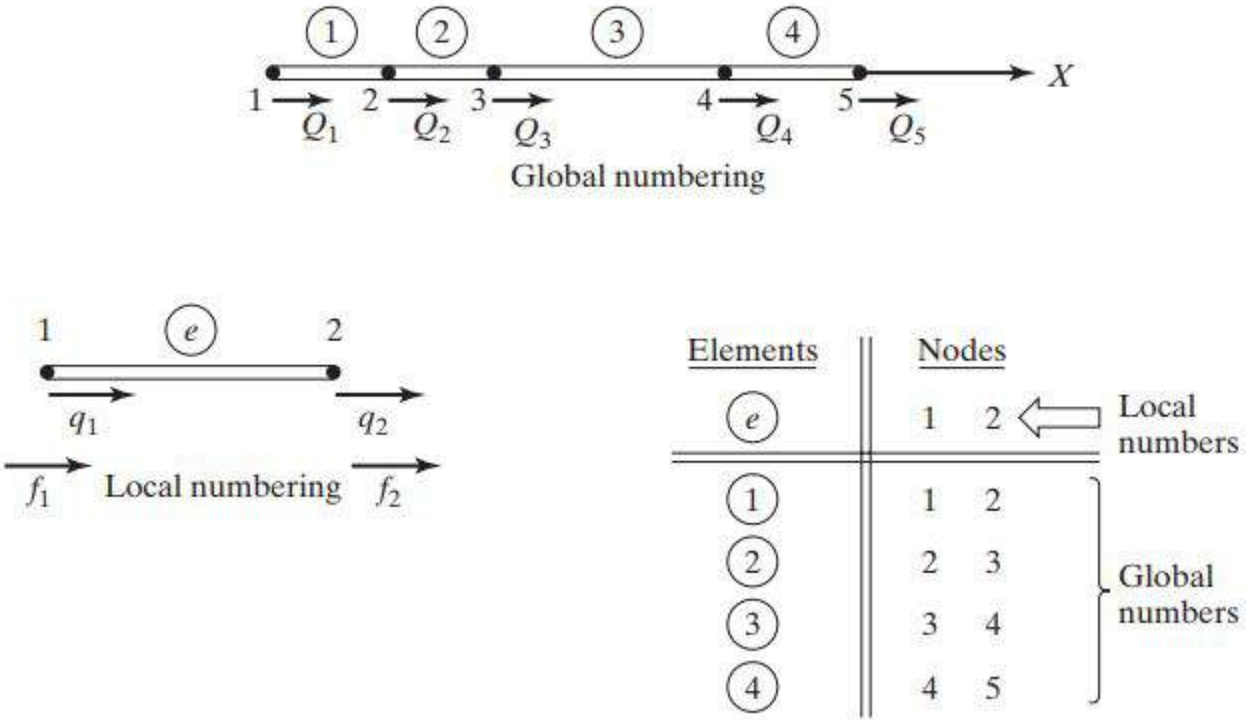
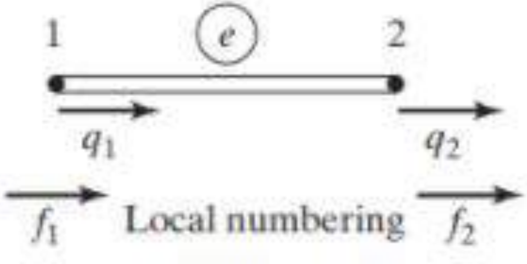


FIGURE 3.4 Element connectivity.

3. Formation of elemental (stiffness matrix and load vector)



$$\mathbf{k}^e = \frac{E_e A_e}{\ell_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\mathbf{q} = [q_1, q_2]^{\mathrm{T}}$$

$$\mathbf{f}^e = \frac{A_e \ell_e f}{2}\begin{Bmatrix} \int_{-1}^{1} N_1 d\xi \\ \int_{-1}^{1} N_2 d\xi \end{Bmatrix}$$

4. Formation of Global (stiffness matrix and load vector)

$$(\mathbf{KQ} - \mathbf{F}) = 0$$

5. Application of boundary condition

Apply displacement and forced boundary conditions

6. Solution of simultaneous equations

$$(\mathbf{KQ} - \mathbf{F}) = 0$$

Interpolation Functions: Interpolation function is also known as approximate function, which is defined to obtain the approximate solution for a given problem, by dividing the domain into smaller elements. It must be selected for each element in such a way that, it must provide better solution for a finite element problem.

Forms of Interpolation Functions

There are two forms of Interpolation functions. They are,

1. Polynomial form
2. Trigonometric form.

Considering one dimensional element in which 'ϕ' represents a field variable and,

The variation of 'ϕ' in polynomial form is represented as,

$$\phi = a_1 + a_2 x + a_3 x^2 + a_4 x^3 + a_5 x^4 + \ldots$$

The variation of 'ϕ' in trigonometric form is represented as,

$$\phi = a_1 \sin\left(\frac{\pi x}{l}\right) + a_2 \sin\left(\frac{3\pi x}{l}\right) + a_3 \sin\left(\frac{5\pi x}{l}\right) + \ldots.$$

Then, consider two equations of 'ϕ',

$$\phi = a_1 + a_2 x + a_3 x^2 + a_4 x^3 + a_5 x^4$$

And, $\phi = a_1 + a_2 x + a_3 x^2$

Equations (1) gives the accurate solution because of its higher order,

For most of the problems, polynomial form is adopted because of the following advantages,

1. Easy to formulate the equation
2. More accurate results are obtained
3. Simple Structure.

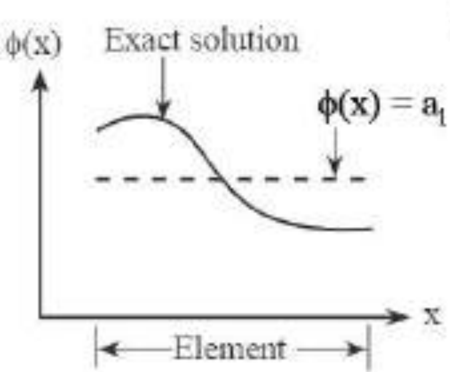


Figure (a): Approximation

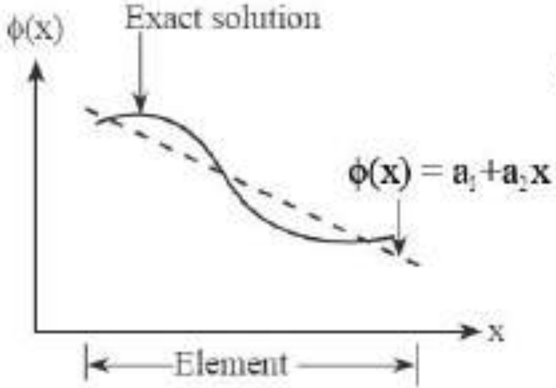


Figure (b): Linear Approximation

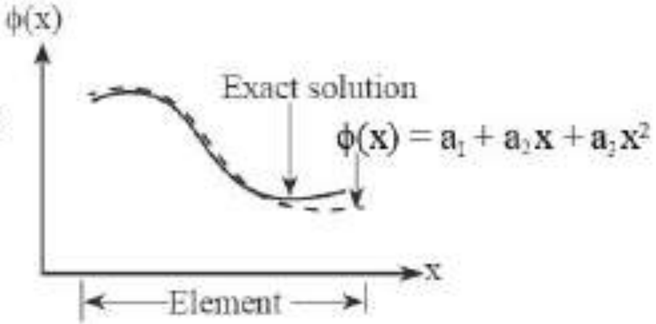
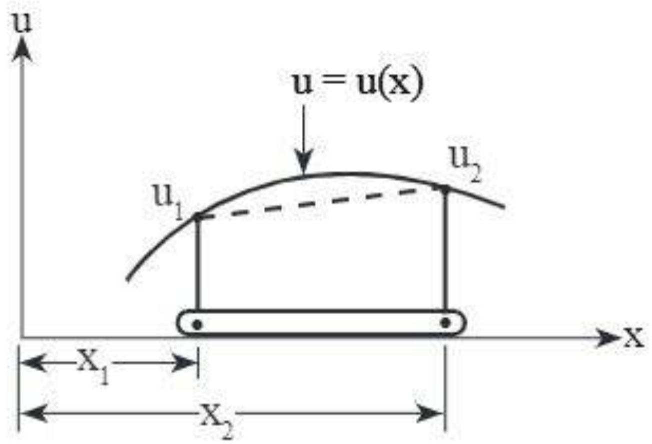


Figure (c): Quadratic Approximation

Derivation of 1D linear interpolation function for the displacement function

Or 2 noded bar element Or Linear bar element



Figure

Function, $u = u(x)$

Consider a linear interpolation formula for a function $u = u(x)$ in the range u_1 and u_2 as,

$$u = a_1 + a_2 x \quad \text{... (1)}$$

Where,

a_1, a_2 – Constants

Applying boundary conditions,

i.e., $u(x_1) = u_1$ and $u(x_2) = u_2$.

Substituting the above values in equation (1),

i.e., $u_1 = a_1 + a_2 x_1$ ... (2)

$u_2 = a_1 + a_2 x_2$ ... (3)

Solving the above two equations,

$$a_1 = \frac{u_1 x_2 - u_2 x_1}{x_2 - x_1} \text{ and}$$

$$a_2 = \frac{u_2 - u_1}{x_2 - x_1}$$

Substituting 'a_1' and 'a_2' values in equation (1),

$$u = \frac{(u_1 x_2 - u_2 x_1)}{(x_2 - x_1)} + x\frac{u_2 - u_1}{(x_2 - x_1)}$$

$$= \frac{u_1 x_2 - u_2 x_1 + u_2 x - u_1 x}{(x_2 - x_1)}$$

$$= \frac{u_1(x_2 - x) + u_2(x - x_1)}{(x_2 - x_1)}$$

$$\therefore \quad \boxed{u = u_1 \frac{(x_2 - x)}{(x_2 - x_1)} + u_2 \frac{(x - x_1)}{(x_2 - x_1)}}$$

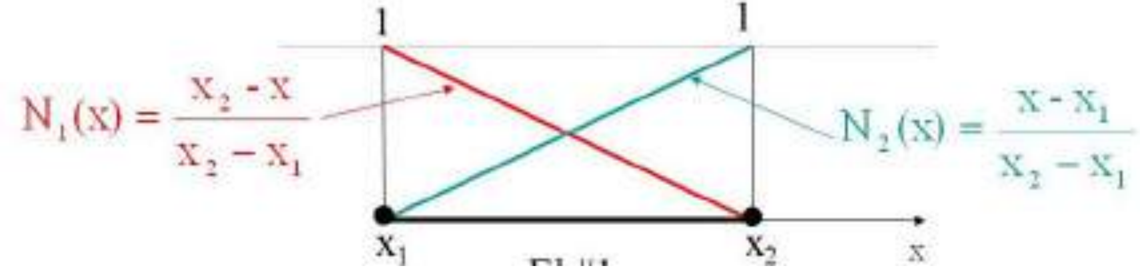
N_1, N_2 – Shape functions

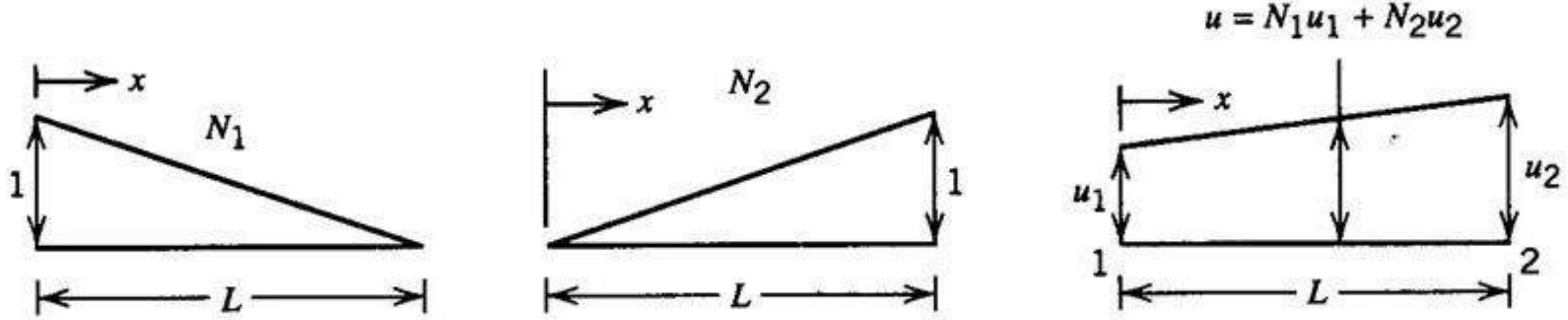
Then, shape functions are given by,

$$\therefore \quad u(x) = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\therefore N_1 = \frac{x_2 - x}{l} = \frac{x_2 - x}{x_2 - x_1} \qquad (\because l = x_2 - x_1)$$

$$\therefore N_2 = \frac{x - x_1}{l} = \frac{x - x_1}{x_2 - x_1}$$





Linear interpolation of the displacement function within an element

Q. Define the shape function. What are the properties of a shape function?

Ans: The mathematical expression which defines the geometry or shape of the finite element is termed as shape function. They are used to determine the variation of field variables such as displacement, temperature, etc. In finite element method, the problems cannot be solved without using shape functions.

The properties of shape functions are as follows,

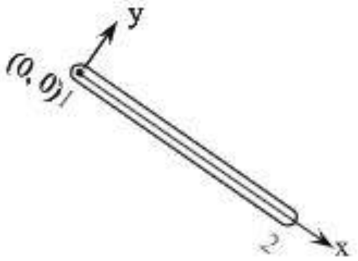
(a) The summation of all the shape functions is equal to 1.

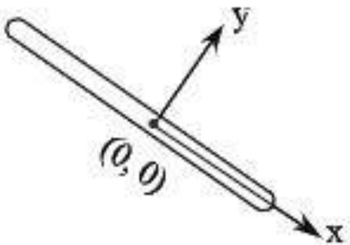
(b) The value of each function at its own node is 1 and the value at other node is zero

(c) The shape functions can be linear or quadratic functions, based on the conditions that, first derivative of shape function should be infinite within the element and the displacements across element boundary should be continuous.

Coordinates, penalty approach

Local Coordinates: In local coordinate system, the nodes of various elements of the structure are specified by the origin, which is placed within the element. This type of coordinate system is adopted, in order to minimize the computational efforts while calculating the global stiffness matrix and displacement vectors. Local coordinates may be different for different elements.





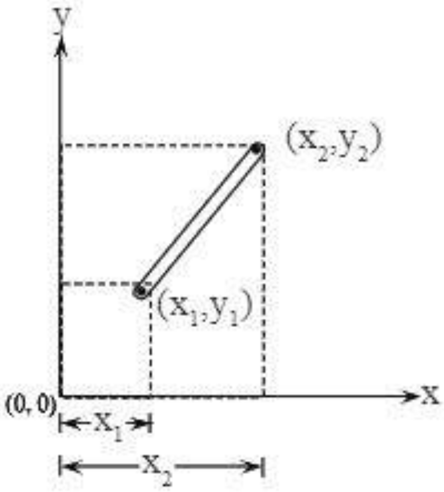
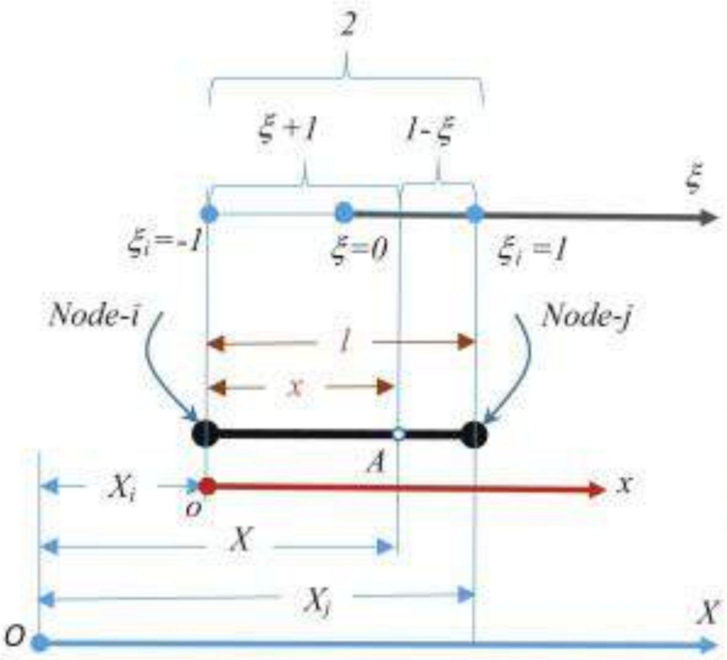


Figure (4): Global Coordinate System

Natural coordinate system



Shape functions:

In natural CS:

$S_i(\xi) = (1-\xi)/2$

$S_j(\xi) = (\xi+1)/2$

In LCS:

$S_i(x) = (l-x)/l$

$S_j(x) = x/l$

In GCS:

$S_i(X) = (X_j-X)/(X_j-X_i)$

$S_j(X) = (X-X_i)/(X_j-X_i)$

GCS – $\{O\text{-}X\}$

LCS– $\{o\text{-}x\}$

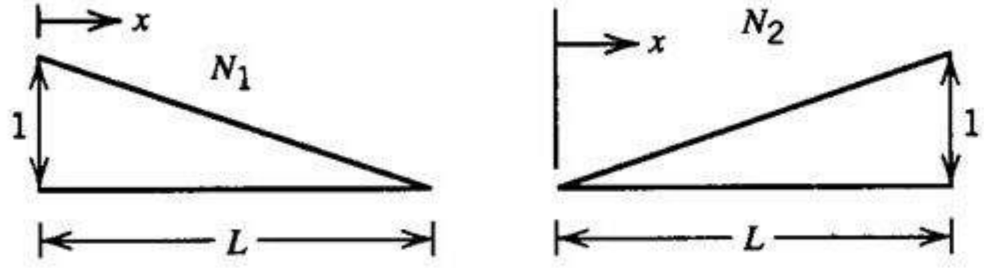
Natural CS – $\{o'\text{-}\ \xi\}$

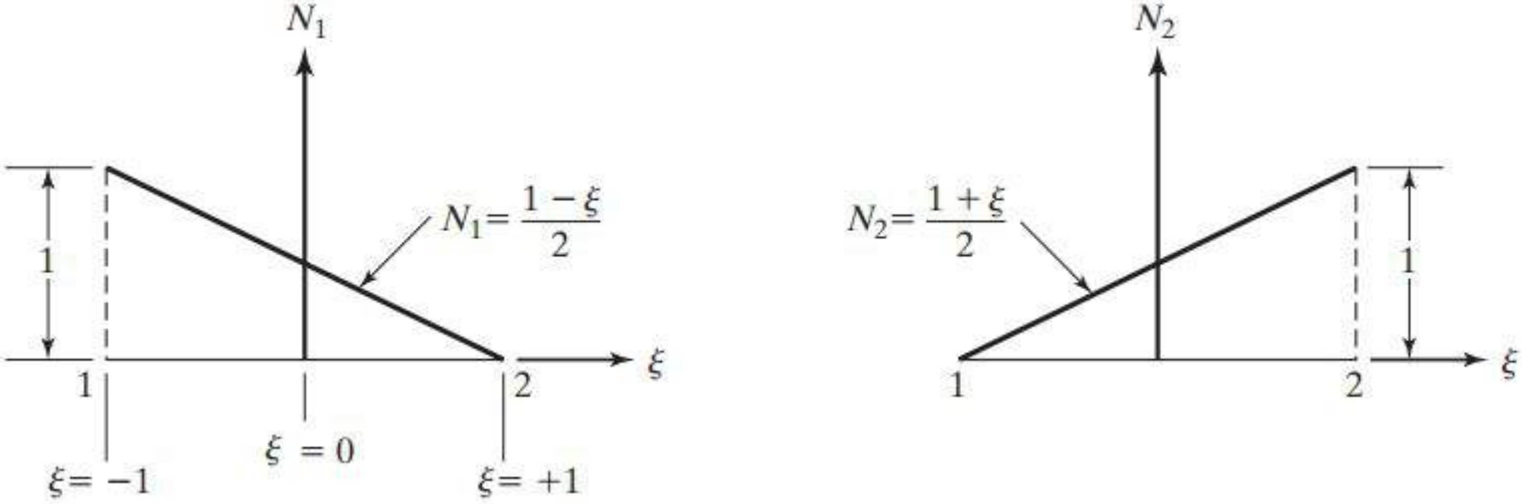
N_1, N_2 – Shape functions

Then, shape functions are given by,

$$\therefore N_1 = \frac{x_2 - x}{l} = \frac{x_2 - x}{x_2 - x_1} \qquad N_1(\xi) = \frac{1-\xi}{2}$$

$$\therefore N_2 = \frac{x - x_1}{l} = \frac{x - x_1}{x_2 - x_1} \qquad N_2(\xi) = \frac{1+\xi}{2}$$





Linear interpolation of the displacement function within an element

Derivation of 1-D Quadratic interpolation function for the displacement function

Or 3 noded bar element Or Quadratic bar element

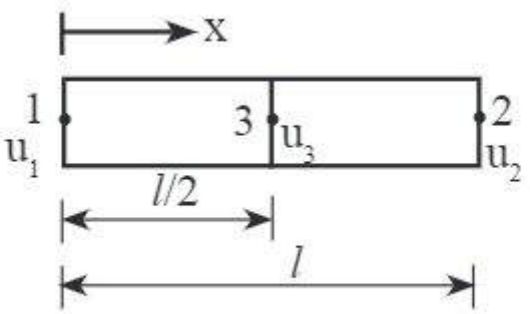


Figure: Quadratic Bar Element

Consider a quadratic bar element of length 'l'. Let u_1, u_2, u_3 are the nodal displacement at nodal points 1,2,3.

The polynomial for one dimensional quadratic bar element is given by,

$$u = a_0 + a_1 x + a_2 x^2 \qquad \dots (1)$$

Boundary Conditions,

At node-1, $u = u_1$, $x_1 = 0$

At node-2, $u = u_2$, $x_2 = l$

At node-3, $u = u_3$, $x_3 = \frac{l}{2}$

Equation (1) can be written in matrix form,

$$\{u\} = \begin{bmatrix} 1 & x & x^2 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \end{Bmatrix} \qquad \dots (2)$$

Substituting the boundary conditions in eq 1. and solving for unknowns a0, a1, a2 we get

$$\{u\} = \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

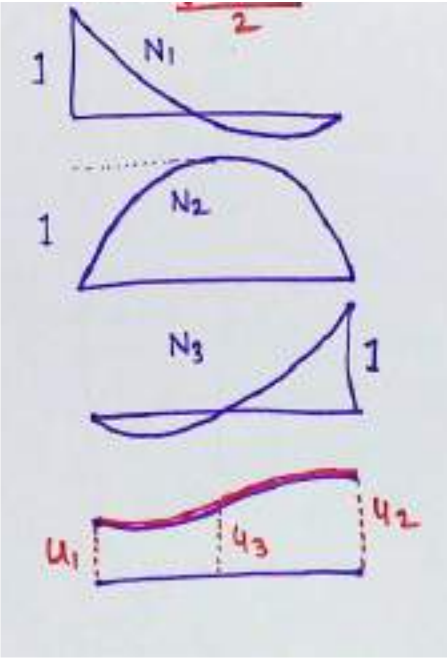
$$\{u\} = N_1 u_1 + N_2 u_2 + N_3 u_3$$

Where,

$$N_1 = 1 - \frac{3x}{l} + \frac{2}{l^2} x^2$$

$$N_2 = \frac{-x}{l} + \frac{2}{l^2} x^2$$

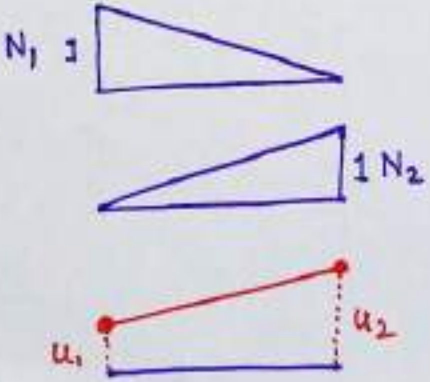
$$N_3 = \frac{4x}{l} - \frac{4}{l^2} x^2$$



Shape Functions for Bar Element

$$N_1 = \frac{1-\xi}{2}$$

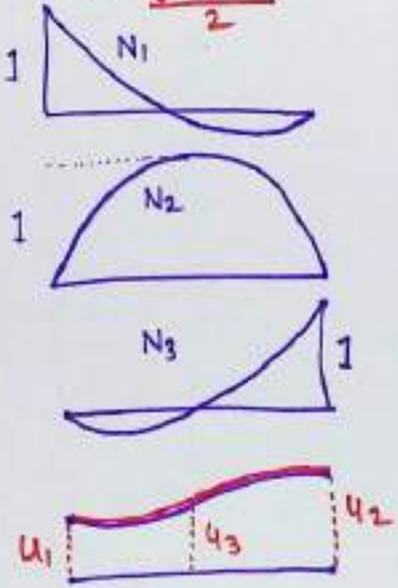
$$N_2 = \frac{1+\xi}{2}$$



$$N_1 = \frac{\xi(\xi-1)}{2}$$

$$N_2 = (1+\xi)(1-\xi)$$

$$N_3 = \frac{\xi(\xi+1)}{2}$$

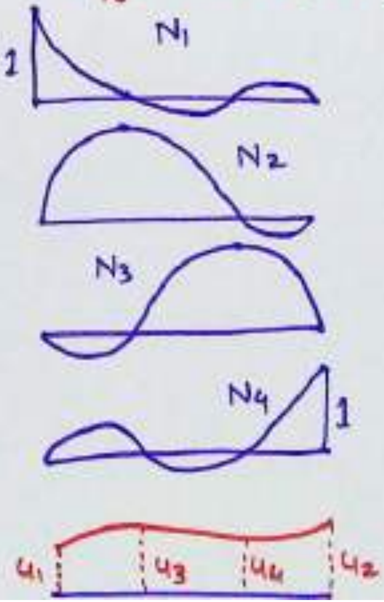


$$N_1 = -\frac{9}{16}\left(\xi+\frac{1}{3}\right)\left(\xi-\frac{1}{3}\right)(\xi-1)$$

$$N_2 = \frac{27}{16}(\xi+1)\left(\xi-\frac{1}{3}\right)(\xi-1)$$

$$N_3 = -\frac{27}{16}(\xi+1)\left(\xi+\frac{1}{3}\right)(\xi-1)$$

$$N_4 = \frac{9}{16}(\xi+1)\left(\xi+\frac{1}{3}\right)\left(\xi-\frac{1}{3}\right)$$



Derivation of strain displacement matrix (using 2 noded bar element)

$$\boxed{\therefore \quad u(x) = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}}$$

$$u = N_1 u_1 + N_2 u_2$$

$$= \left(\frac{x_2 - x}{l} \right) u_1 + \left(\frac{x - x_1}{l} \right) u_2$$

The strain for element is defined as,

$$\varepsilon = \frac{du}{dx} = \frac{u_2 - u_1}{l}$$

Also,

$$\varepsilon = \frac{du}{dx} = \frac{dN_1}{dx} u_1 + \frac{dN_2}{dx} u_2$$

$$\varepsilon = \frac{du}{dx} = \frac{dN_1}{dx}u_1 + \frac{dN_2}{dx}u_2$$

$$= \frac{-1}{l}u_1 + \frac{1}{l}u_2$$

$$= \frac{1}{l}\,[-1 \;\; 1]\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\varepsilon = \frac{1}{(x_2 - x_1)}[-1 \;\; 1]\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

i.e., $\varepsilon = [B]\ \{u\}$

Where,

$$[B] = \frac{1}{(x_2 - x_1)}[-1 \;\; 1]$$

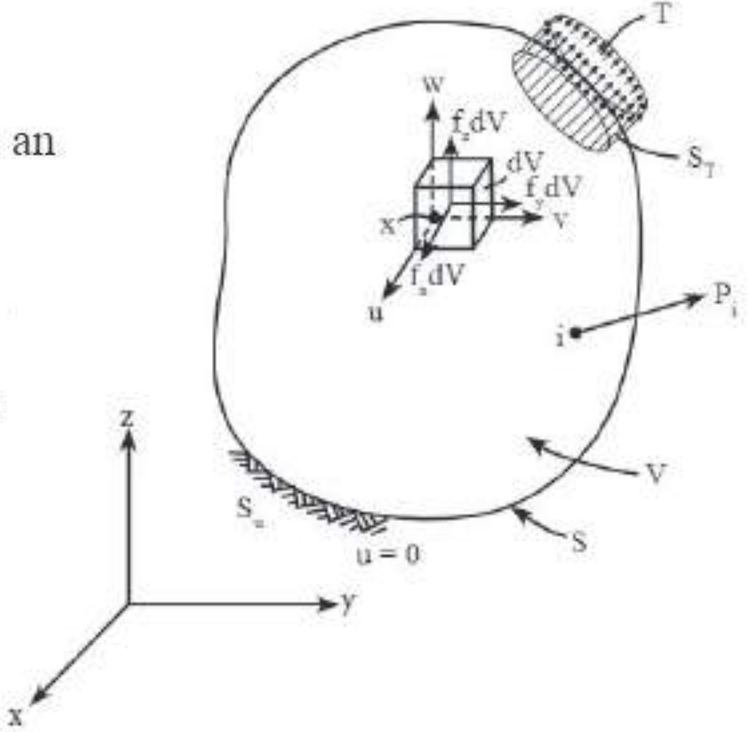
$$\therefore [B] = \frac{1}{l}\,[-1 \;\; 1]$$

The above equation is known as 'element strain displacement matrix' for one dimensional element.

Potential Energy Approach

General expression for total potential energy in an elastically loaded structure is,

$$\pi = \frac{1}{2}\int_V \sigma^T \varepsilon dV - \int_V u^T f dv - \int_A u^T \mathbf{T} dA - \sum_i u_i^T P_i$$



It can be written for the 1D problems as

$$\Pi = \frac{1}{2}\int_L \sigma^{\mathrm{T}} \epsilon A \, dx - \int_L u^{\mathrm{T}} f A \, dx - \int_L u^{\mathrm{T}} T \, dx - \sum_i u_i P_i$$

Derivation of Elemental stiffness matrix

Consider first term in the general expression,

Total strain energy,

$$U_e = \frac{1}{2}\int_V \sigma^T \varepsilon dV$$

$$U_e = \frac{1}{2}\int_V (EBu)^T (Bu) A dx$$

$$[\because \sigma = EBu \text{ and } \varepsilon = Bu]$$

$$= \frac{1}{2} u^T u E_e A_e B^T B \int_{x_1}^{x_2} dx \qquad \ldots (1)$$

Element Stiffness Matrix, $K^{(e)}$

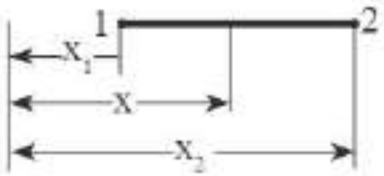


Figure (1)

Let, x_1, x_2 – Lengths at node-1 and node-2

u_1, u_2 – Displacement vectors

Where,

B – Strain displacement matrix

E_e – Young's modulus of the element

A_e – Cross-sectional area of the element

And,

$$B = \frac{1}{x_2 - x_1}\begin{bmatrix} -1 & 1 \end{bmatrix}$$

$$B = \frac{1}{l_e}\begin{bmatrix} -1 & 1 \end{bmatrix} \qquad [\because \; x_2 - x_1 = l_e]$$

$$B^T = \frac{1}{l_e}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$B^T B = \frac{1}{l_e}\begin{bmatrix} -1 \\ 1 \end{bmatrix}\frac{1}{l_e}\begin{bmatrix} -1 & 1 \end{bmatrix}$$

$$B^T B = \frac{1}{l_e^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

And $\int_{x_1}^{x_2} dx = [x]_{x_1}^{x_2} = x_2 - x_1 = l_e$

Where,

l_e – Length of the element

We have the equation----1.

$$U_e = \frac{1}{2} u^T u E_e A_e B^T B \int_{x_1}^{x_2} dx$$

$\therefore$ On substituting in equation (1),

$$U_e = \frac{1}{2}u^T\left\{E_e A_e \frac{1}{l_e^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} l_e\right\}u$$

$$U_e = \frac{1}{2}u^T K_e u$$

Where,

Stiffness matrix, $K_e = \frac{E_e A_e}{l_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

And, global stiffness matrix $K_{\text{global}} = \sum_e K_e$ i.e., summation of individual element stiffness matrices.

Element Body Load Vector (F_e)

Considering the second term in π_T.

$$\int_v u^T f dv = \int_l (Nu)^T fA dx$$

$$= \int_l u^T N^T fA dx$$

$$= u^T A_e f \int_l N^T dx = u^T A_e f \begin{Bmatrix} \int N_1 dx \\ \int N_2 dx \end{Bmatrix}$$

$$= u^T \left[\frac{A_e f l_e}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \right] \quad \left[\because \int N_1 dx = \int N_2 dx = \frac{l_e}{2} \right]$$

$$= u^T . F_e$$

Where,

$$F_e = \frac{f A_e l_e}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

Element Traction Load Vector (T_e)

Considering the third term in π_T.

$$\int_l u^T T dx = \int_l (Nu)^T T dx = u^T\{T \int N^T dx\}$$

$$= u^T T \begin{Bmatrix} \int N_1 dx \\ \int N_2 dx \end{Bmatrix}$$

$$\int_l u^T T dx = u^T \left[T \frac{l_e}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \right] = u^T T_e$$

Where,

$$\because T_e = \frac{T l_e}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

Temperature Load Vector

Thermal load= A * thermal stress

If a temperature gradient exists then, temperature load,

$$\theta = AE\varepsilon_0$$

$$= AE\,\alpha\,\Delta T$$

Temperature load vector,

$$\theta_e = AE\,\alpha\,\Delta T \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \begin{matrix} 1 \\ 2 \end{matrix}$$

Then, global load vector,

$$F = \sum_e [F_e + T_e + \theta_e] + P_i$$

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3.10 TEMPERATURE EFFECTS

$$U = \int_L \frac{1}{2}(\epsilon - \epsilon_0)^{\mathrm{T}} E(\epsilon - \epsilon_0) A\,dx$$

Element stiffness matrix,

$$K_e = \frac{E_e A_e}{l_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

global stiffness matrix $K_{\text{global}} = \sum_e K_e$

K (global) U = F (global)

Global load vector,

$$F = \sum_e (F_e + T_e + \theta_e) + P_i$$

Where,

$$F_e = \frac{f A_e l_e}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$\because T_e = \frac{T l_e}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$\theta_e = A_e E_e \alpha \Delta T \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

Equilibrium Equation

$$[K]\{u\} = \{F\}$$

Where,

$[K]$ – Global stiffness matrix

$\{u\}$ – Global displacement vector

$\{F\}$ – Global load vector

Equation for stress, strain and support reactions are,

$$\sigma = E\varepsilon, \quad \varepsilon = Bu, \quad R = Ku = F$$

If temperature gradient is present then,

$$\sigma = E(\varepsilon - \varepsilon_0)$$

$$= E[\varepsilon - \alpha\Delta T]$$

$$\sigma = E[Bu - \alpha\Delta T]$$

Example

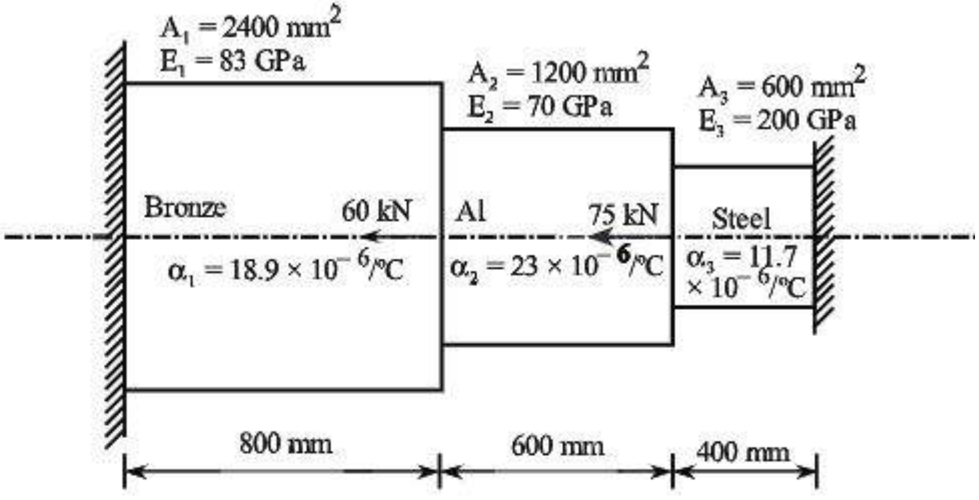
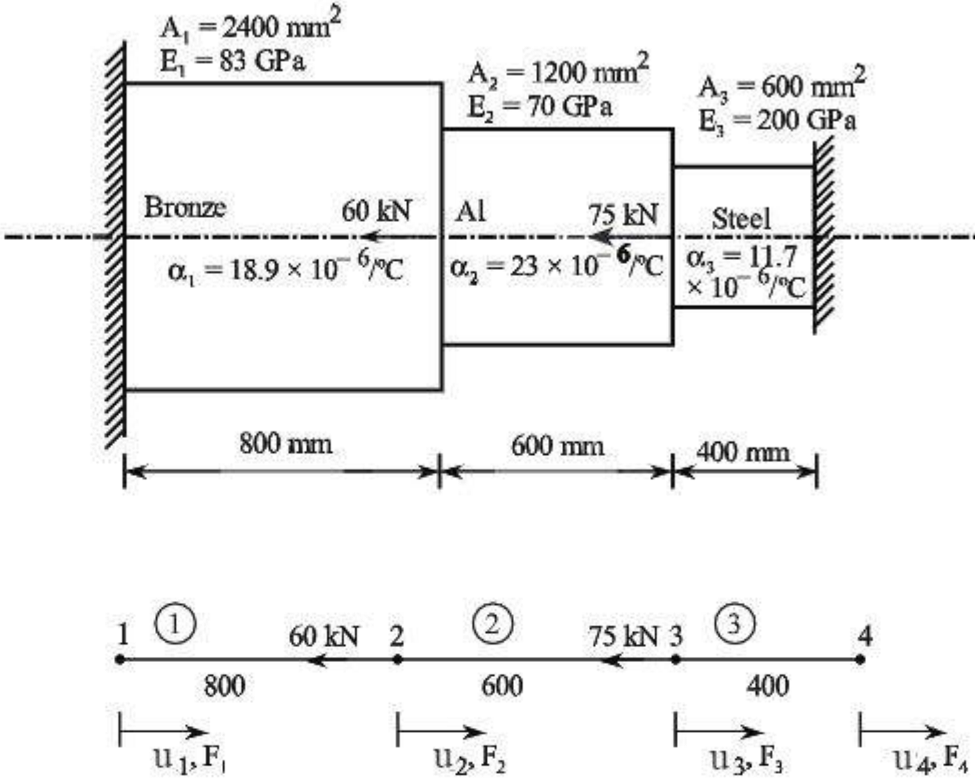


Figure (2)

Temperature gradient, $\Delta T = 80°C$

For this problem, calculate nodal displacements, stresses in each bar, Reactions at the supports



Boundary conditions, $u_1 = u_4 = 0$

Basic Steps Involved In FEM:

1. Domain Discretization
2. Selection of displacement functions
3. Formation of elemental (stiffness matrix and load vector)
4. Formation of Global (stiffness matrix and load vector) : K U = F
5. Application of boundary condition
6. Solution of simultaneous equations (for unknown nodal displacements)
7. Calculation of stresses and strains
8. Interpretation of results

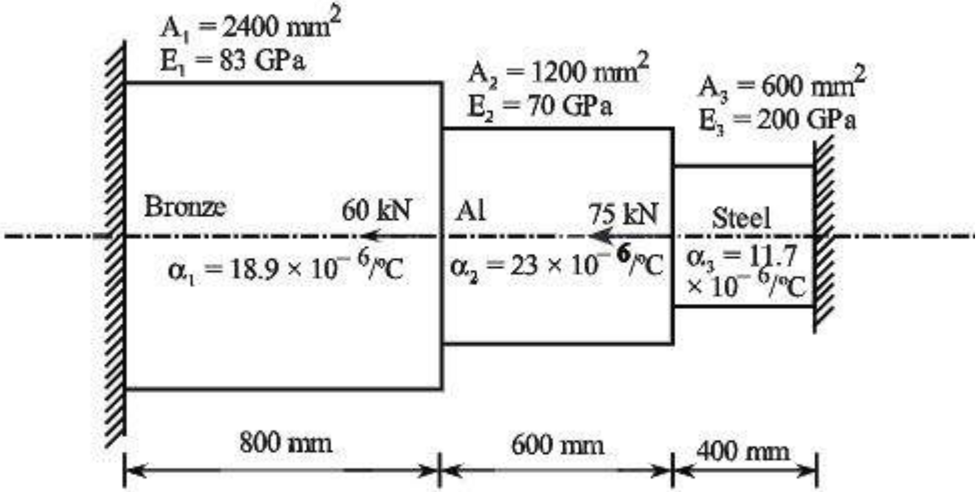
Element stiffness matrix,

$$K_e = \frac{E_e A_e}{l_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Stiffness matrix of element–1,

$$K_1 = \frac{83 \times 10^3 \times 2400}{800}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$K_1 = 10^5 \begin{array}{c} \begin{array}{cc} 1 & \quad 2 \end{array} \\ \begin{bmatrix} 2.49 & -2.49 \\ -2.49 & 2.49 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \end{array}$$



Stiffness matrix of element–2,

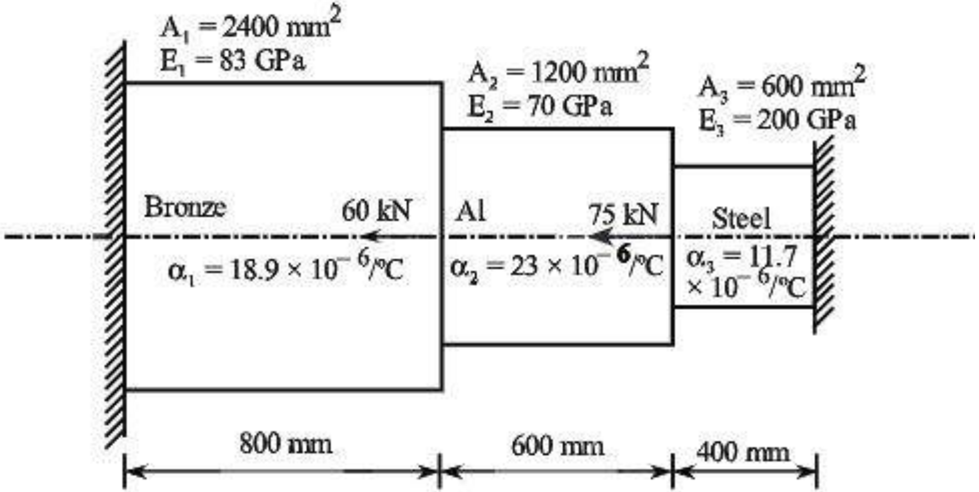
$$K_2 = \frac{70 \times 10^3 \times 1200}{600} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$K_2 = 10^5 \begin{array}{c} \begin{array}{cc} 2 & 3 \end{array} \\ \begin{bmatrix} 1.4 & -1.4 \\ -1.4 & 1.4 \end{bmatrix} \end{array} \begin{array}{c} \\ 2 \\ 3 \end{array}$$

Stiffness matrix of element–3,

$$K_3 = \frac{200 \times 10^3 \times 600}{400} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$K_3 = 10^5 \begin{array}{c} \begin{array}{cc} 3 & 4 \end{array} \\ \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix} \end{array} \begin{array}{c} \\ 3 \\ 4 \end{array}$$



Global stiffness matrix,

$$K = 10^5 \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{cccc} 2.49 & -2.49 & 0 & 0 \\ -2.49 & 2.49+1.4 & -1.4 & 0 \\ 0 & -1.4 & 1.4+3 & -3 \\ 0 & 0 & -3 & 3 \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}$$

$$= 10^5 \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{cccc} 2.49 & -2.49 & 0 & 0 \\ -2.49 & 3.89 & -1.4 & 0 \\ 0 & -1.4 & 4.4 & -3 \\ 0 & 0 & -3 & 3 \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}$$

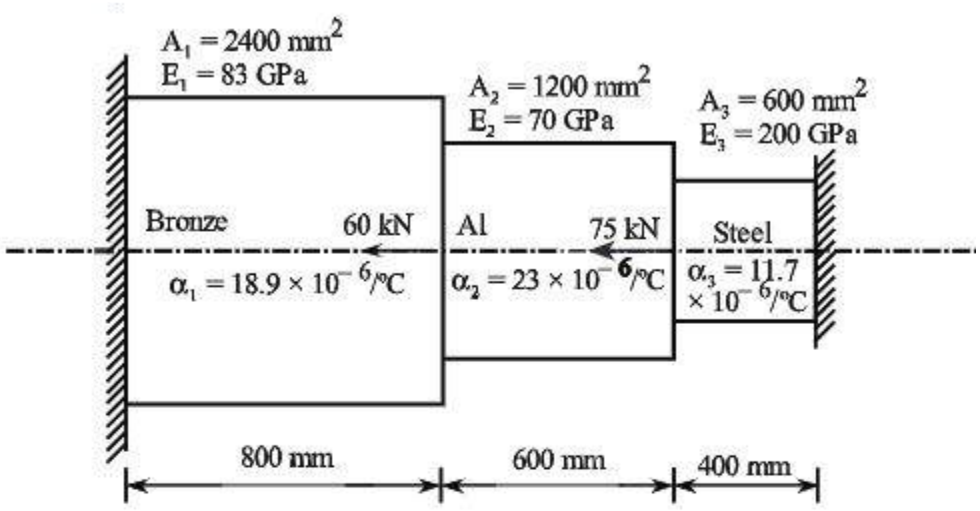
Global load vector,

$$F = \sum_e (F_e + T_e + \theta_e) + P_i$$

In this problem, body loads and traction loads are absent.

Temperature loads,

$$\theta_e = A_e E_e \alpha \Delta T \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$



Temperature load on element–1,

$$\theta_1 = 2400 \times 83 \times 10^3 \times 18.9 \times 10^{-6} \times 80 \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

$$\theta_1 = 10^5 \begin{Bmatrix} -3.012 \\ 3.012 \end{Bmatrix} \begin{matrix} 1 \\ 2 \end{matrix}$$

Temperature load on element–2,

$$\theta_2 = 1200 \times 70 \times 10^3 \times 23 \times 10^{-6} \times 80 \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

$$\theta_2 = 10^5 \begin{Bmatrix} -1.54 \\ 1.54 \end{Bmatrix} \begin{matrix} 2 \\ 3 \end{matrix}$$

Temperature load on element–3,

$$\theta_3 = 600 \times 200 \times 10^3 \times 11.7 \times 10^{-6} \times 80 \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

$$\theta_3 = 10^5 \begin{Bmatrix} -1.12 \\ 1.12 \end{Bmatrix} \begin{matrix} 3 \\ 4 \end{matrix}$$

Points loads,

At node-1 = 0

At node-2 = -60×10^3 N

At node-3 = -75×10^3 N

At node-4 = 0

Global load vector,

$$F = \sum_e (F_e + T_e + \theta_e) + P_i$$

$$\therefore \qquad \{F\} = \begin{array}{l} \begin{Bmatrix} -3.012 \times 10^5 \\ (3.012 - 1.54 - 0.6) \times 10^5 \\ (1.54 - 1.12 - 0.75) \times 10^5 \\ 1.12 \times 10^5 \end{Bmatrix} \end{array} \begin{array}{l} 1 \\ 2 \\ 3 \\ 4 \end{array}$$

$$\{F\} = 10^5 \begin{Bmatrix} -3.012 \\ 0.872 \\ -0.33 \\ 1.12 \end{Bmatrix} \begin{array}{l} 1 \\ 2 \\ 3 \\ 4 \end{array}$$

Then,

$[K]\{u\} = \{F\}$

$$10^5 \begin{bmatrix} 2.49 & -2.49 & 0 & 0 \\ -2.49 & 3.89 & -1.4 & 0 \\ 0 & -1.4 & 4.4 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = 10^5 \begin{Bmatrix} -3.012 \\ 0.872 \\ -0.33 \\ 1.12 \end{Bmatrix}$$

Applying the boundary conditions, $u_1 = u_4 = 0$.

By elimination approach, the above equation reduces to,

$$10^5 \begin{Bmatrix} 3.89 & -1.4 \\ -1.4 & 4.4 \end{Bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = 10^5 \begin{Bmatrix} 0.872 \\ -0.33 \end{Bmatrix}$$

$$3.89\, u_2 - 1.4\, u_3 = 0.872$$

$$-1.4\, u_2 + 4.4\, u_3 = -0.33$$

Solving the above two equations,

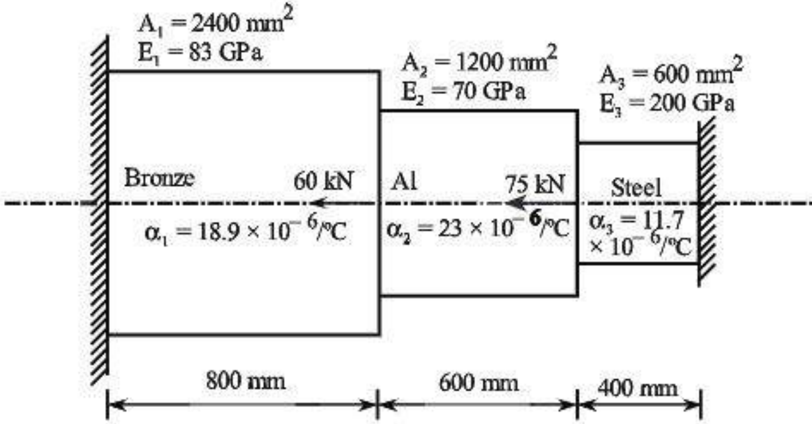
$$u_2 = 0.223 \text{ mm}$$

$$u_3 = -0.00415 \text{ mm}$$

$\therefore$ Global displacement vector, $u = [0 \;\; 0.223 \;\; -0.00415 \;\; 0]^T$ mm

Calculation of stresses,

$$\sigma = E(Bu - \alpha \Delta T)$$



Stress induced in element–1,

$$\sigma_1 = 83 \times 10^3 \left[\frac{1}{800} [-1 \ \ 1] \begin{bmatrix} 0 \\ 0.223 \end{bmatrix} - 18.9 \times 10^{-6} \times 80 \right]$$

$\therefore \ \sigma_1 = -102.366$ MPa

Stress induced in element–2,

$$\sigma_2 = 70 \times 10^3 \left[\frac{1}{600} [-1 \ \ 1] \begin{bmatrix} 0.223 \\ -0.004 \end{bmatrix} - 23 \times 10^{-6} \times 80 \right]$$

$\therefore \ \sigma_2 = -155.28$ MPa

Stress induced in element–3,

$$\sigma_3 = 200 \times 10^3 \left[\frac{1}{400} [-1 \ \ 1] \begin{bmatrix} -0.004 \\ 0 \end{bmatrix} - 11.7 \times 10^{-6} \times 80 \right]$$

$\therefore \ \sigma_3 = -185.2$ MPa

Solving for the support reactions,

$$K11\ u1 + K_{12}u_2 + K_{13}u_3 + K14\ u4 = F_1 + R_1$$

$$K41\ u1 + K_{42}u_2 + K_{43}u_3 + K44\ u4 = F_4 + R_4$$

$$(-2.49 \times 0.223) \times 10^5 = R_1 + (-3.012) \times 10^5$$

$$\therefore \quad R_1 = 245.6 \text{ kN}$$

$$(-3 \times -0.004) \times 10^5 = R_4 + (1.12 \times 10^5)$$

$$\therefore \quad R_4 = -110.8 \text{ kN}$$

Example 3.8

An axial load $P = 300 \times 10^3$ N is applied at 20°C to the rod as shown in Fig. E3.8. The temperature is then raised to 60°C.

(a) Assemble the **K** and **F** matrices.

(b) Determine the nodal displacements and element stresses.

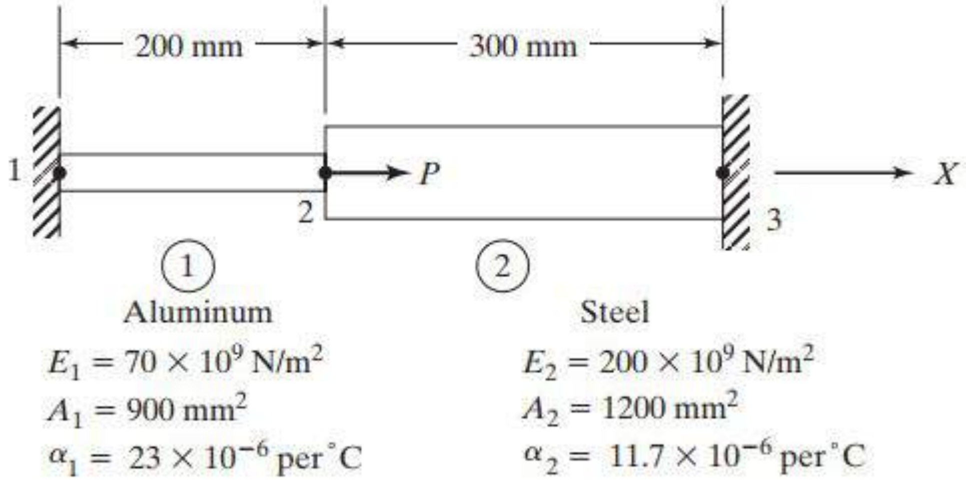


FIGURE E3.8

Solution

(a) The element stiffness matrices are

$$\mathbf{k}^1 = \frac{70 \times 10^3 \times 900}{200}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \text{N/mm}$$

$$\mathbf{k}^2 = \frac{200 \times 10^3 \times 1200}{300}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \text{N/mm}$$

Thus,

$$\mathbf{K} = 10^3\begin{bmatrix} 315 & -315 & 0 \\ -315 & 1115 & -800 \\ 0 & -800 & 800 \end{bmatrix} \text{N/mm}$$

Now, in assembling $\mathbf{F}$, both temperature and point load effects have to be considered. The element temperature forces due to $\Delta T = 40°\text{C}$ are obtained from Eq. 3.106b as

$$\boldsymbol{\Theta}^1 = 70 \times 10^3 \times 900 \times 23 \times 10^{-6} \times 40 \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \begin{matrix} \downarrow \text{Global dof} \\ 1 \\ 2 \end{matrix} \ \text{N}$$

and

$$\boldsymbol{\Theta}^2 = 200 \times 10^3 \times 1200 \times 11.7 \times 10^{-6} \times 40 \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \begin{matrix} 2 \\ 3 \end{matrix} \ \text{N}$$

Upon assembling $\boldsymbol{\Theta}^1$, $\boldsymbol{\Theta}^2$, and the point load, we get

$$\mathbf{F} = 10^3 \begin{Bmatrix} -57.96 \\ 57.96 \quad - 112.32 \ + 300 \\ 112.32 \end{Bmatrix}$$

or

$$\mathbf{F} = 10^3[-57.96, \quad 245.64, \quad 112.32]^{\mathrm{T}} \ \text{N}$$

(b) The elimination approach will now be used to solve for the displacements. Since dof 1 and 3 are fixed, the first and third rows and columns of **K**, together with the first and third components of **F**, are deleted. This results in the scalar equation

$$10^3[1115]Q_2 = 10^3 \times 245.64$$

yielding

$$Q_2 = 0.220 \text{ mm}$$

Thus,

$$\mathbf{Q} = [0, \quad 0.220, \quad 0]^{\mathrm{T}} \text{ mm}$$

In evaluating element stresses, we have to use Eq. 3.108b:

$$\begin{aligned}\sigma_1 &= \frac{70 \times 10^3}{200}[-1 \quad 1]\begin{Bmatrix} 0 \\ 0.220 \end{Bmatrix} - 70 \times 10^3 \times 23 \times 10^{-6} \times 40 \\ &= 12.60 \text{ MPa}\end{aligned}$$

and

$$\begin{aligned}\sigma_2 &= \frac{200 \times 10^3}{300}[-1 \quad 1]\begin{Bmatrix} 0.220 \\ 0 \end{Bmatrix} - 200 \times 10^3 \times 11.7 \times 10^{-6} \times 40 \\ &= -240.27 \text{ MPa}\end{aligned}$$

■

Concept of assembly

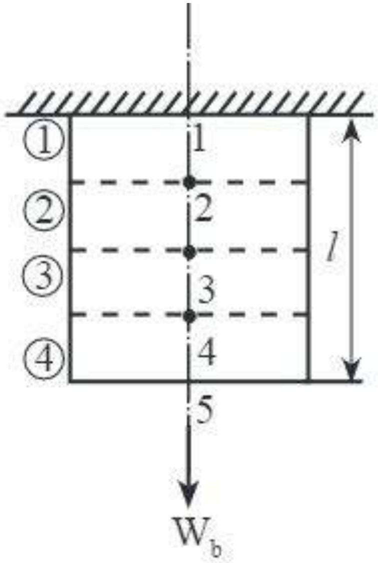


Figure (1): Bar Element

[K] { U } = { F }

$$
\frac{AE}{l}
\begin{array}{c}
\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\
\begin{bmatrix}
1 & -1 & 0 & 0 & 0 \\
-1 & 1+1 & -1 & 0 & 0 \\
0 & -1 & 1+1 & -1 & 0 \\
0 & 0 & -1 & 1+1 & -1 \\
0 & 0 & 0 & -1 & 1
\end{bmatrix}
\end{array}
\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}
\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix}
=
\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \end{Bmatrix}
$$

$$
\frac{AE}{l}
\begin{array}{c}
\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\
\begin{bmatrix}
1 & -1 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 \\
0 & 0 & -1 & 2 & -1 \\
0 & 0 & 0 & -1 & 1
\end{bmatrix}
\end{array}
\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}
\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix}
=
\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \end{Bmatrix}
$$

Using two finite elements, find the stress distribution in a uniformly tapering bar of cross sectional area 300 mm² and 200 mm² at their ends, length 100 mm, subjected to an axial tensile load of 50 N at smaller end and fixed at larger end. Take E = 2 × 10⁵ N/mm².

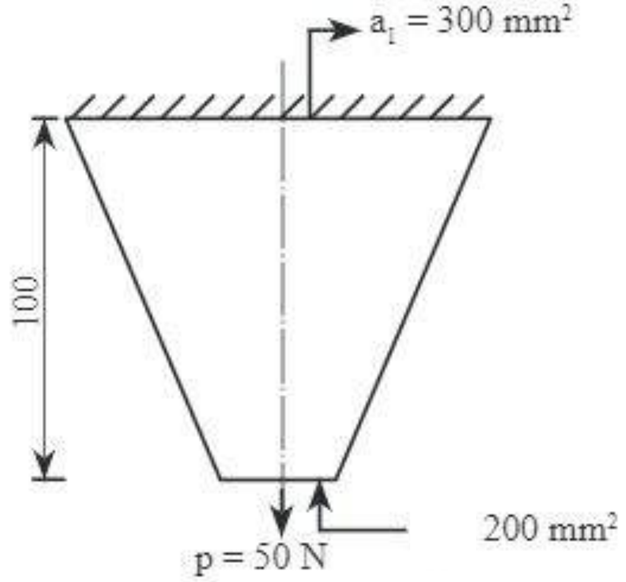


Figure (1): Tapered Bar

Given that,

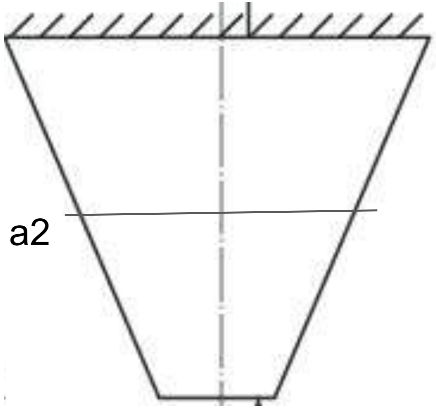
Cross sectional area of bigger end, $a_1 = 300\ mm^2$

Cross sectional area of smaller end, $a_3 = 200\ mm^2$

Length of the bar, $l = 100$ mm

Axial tensile load, $P = 50$ N

Young's modulus, $E = 2 \times 10^5\ N/mm^2$



$$a_1 = 300\ mm^2$$

$$a_3 = 200\ mm^2$$

$$a_2 = \frac{a_1 + a_3}{2}$$

$$= \frac{300 + 200}{2}$$

$$a_2 = 250\ mm^2$$

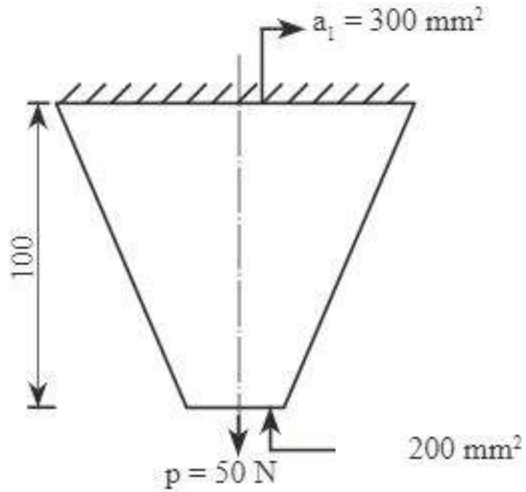


Figure (1): Tapered Bar

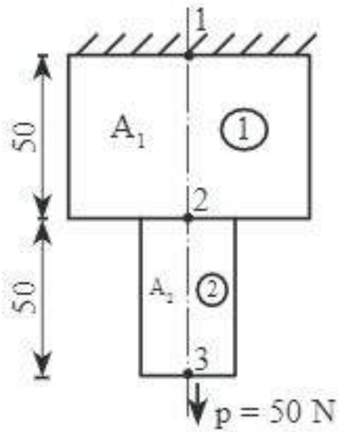


Figure (2): Stepped Bar

Areas of the Stepped Bar

Area at node 1, $a_1 = 300 \text{ mm}^2$

Area at node 3, $a_3 = 200 \text{ mm}^2$

Area at node 2, $a_2 = \dfrac{a_1 + a_3}{2}$

$$= \frac{300 + 200}{2}$$

$$a_2 = 250 \text{ mm}^2$$

Area at element (1), $A_1 = \dfrac{a_1 + a_2}{2}$

$$= \frac{300 + 250}{2}$$

$$A_1 = 275 \text{ mm}^2$$

Area of element (2), $A_2 = \dfrac{a_2 + a_3}{2}$

$$= \frac{200 + 250}{2}$$

$$A_2 = 225 \text{ mm}^2$$

Element (1)

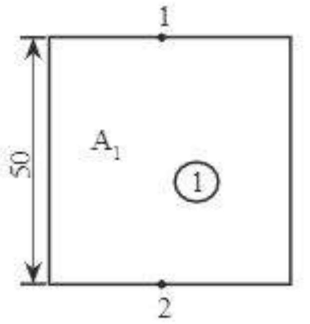


Figure (3): Element (1)

Finite element equation for element (1) is given by,

$$\frac{A_1E_1}{l_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$\frac{275 \times 2 \times 10^5}{50}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$11 \times 10^5\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$10^5 \begin{array}{cc} \begin{array}{cc} 1 & 2 \end{array} & \\ \begin{bmatrix} 11 & -11 \\ -11 & 11 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \end{array} \end{array} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \qquad \ldots (1)$$

Element (2)

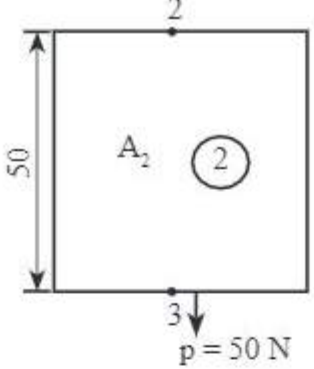


Figure (4): Element (2)

Finite element equation for element (2) is given by,

$$\frac{A_2E_2}{l_2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix}$$

$$\frac{225 \times 2 \times 10^5}{50}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix}$$

$$9 \times 10^5\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix}$$

$$10^5 \begin{array}{cc} \begin{array}{cc} 2 & 3 \end{array} & \\ \begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix} & \begin{array}{c} 2 \\ 3 \end{array} \end{array} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix} \qquad \ldots (2)$$

From equation (1) and (2),

Finite element equation is given by,

$$\begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ 10^5 & \begin{bmatrix} 11 & -11 & 0 \\ -11 & 11+9 & 9 \\ 0 & -9 & 9 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

$$10^5 \begin{bmatrix} 11 & -11 & 0 \\ -11 & 20 & -9 \\ 0 & -9 & 9 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} \quad \ldots (3)$$

Applying the boundary conditions,

$$u_1 = 0, F_1 = 0, F_2 = 0, F_3 = 50\text{N}$$

From equation (3),

$$10^5 \begin{bmatrix} 11 & -11 & 0 \\ -11 & 20 & -9 \\ 0 & -9 & 9 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 50 \end{Bmatrix}$$

Eliminating the first row and first column,

Since $u_1 = 0$

$$10^5 \begin{bmatrix} 20 & -9 \\ -9 & 9 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 50 \end{Bmatrix}$$

$$10^5(20u_2 - 9u_3) = 0$$

$$10^5(-9u_2 + 9u_3) = 50$$

On solving the above equations,

$$u_2 = 4.545 \times 10^{-5}$$

$$u_3 = 10.101 \times 10^{-5}$$

Stress Distribution

For element (1)

$$\text{Stress, } \sigma_1 = E_1 \times \frac{u_2 - u_1}{l_1}$$

$$= 2 \times 10^5 \times \frac{(4.545 \times 10^{-5} - 0)}{50}$$

$$\sigma_1 = 0.1818 \text{ N/mm}^2$$

For element (2)

$$\text{Stress, } \sigma_2 = E_2 \times \frac{u_3 - u_2}{l_2}$$

$$= 2 \times 10^5 \times \frac{(10.10 \times 10^{-5} - 4.545 \times 10^{-5})}{50}$$

$$\sigma_2 = 0.222 \text{ N/mm}^2$$

What is study state heat transfer analysis? Write its governing Equation?

Steady state heat transfer is defined as the temperature at any point in the medium does not change with time.

For a one dimensional steady state heat transfer,

$$K.\frac{d^2T}{dx^2} + q = 0$$

K – Thermal conductivity

T – Temperature

q – Internal heat source per unit volume

3D Conduction heat transfer

General 3D conduction Equation:

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \dot{q} = \rho c\frac{\partial T}{\partial \tau}$$

For constant conductivity:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial \tau}$$

$$\alpha = k/\rho c$$

= Thermal diffusivity of a material

Q. Give the finite element equation for a one dimensional heat conduction element.

Ans: The finite element equation for a one dimensional heat conduction element is given by,

$$\{F\} = [K_c]\{T\}$$

$\{F\}$ – Force vector

$$= \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \text{ for a two noded element}$$

$[K_c]$ – Stiffness matrix in case of heat conduction

$$= \frac{KA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$\{T\}$ – Nodal temperature vector

$$= \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} \text{ for a two noded element}$$

Similar to structural problems

- Formulation of element stiffness matrices
- Assembly of global stiffness matrix [K]
- formulation of load vector {F}
- Solution of [K]{u}={F} to get nodal displacements {u}

Unit 2

Trusses

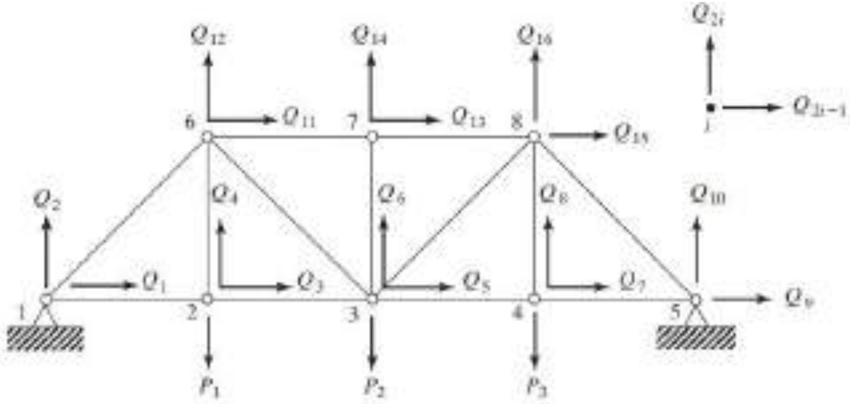
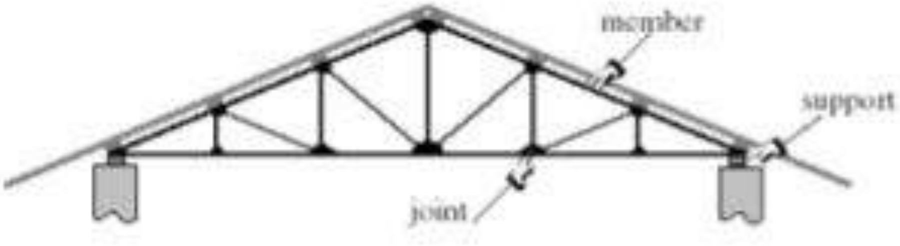
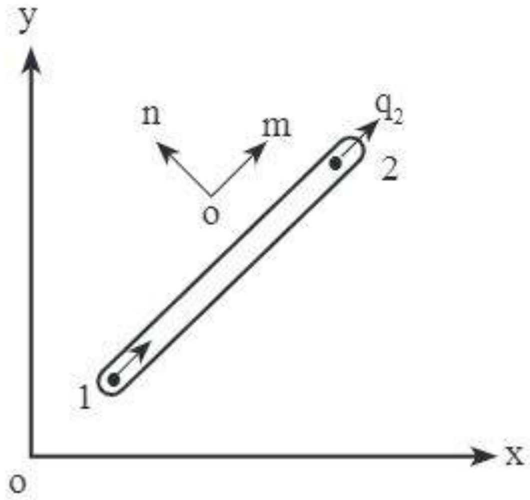
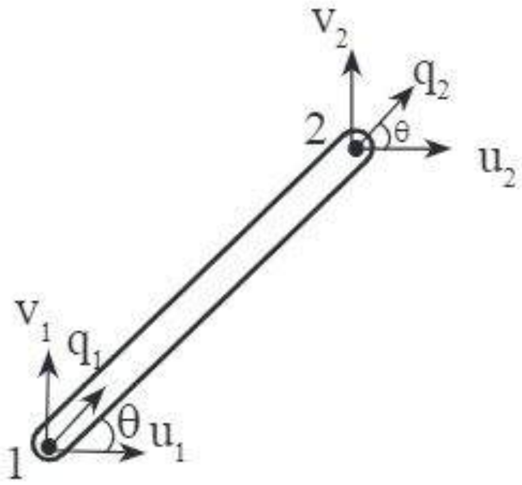


FIGURE 4.1 A two-dimensional truss.





Figure(1): Pin Jointed Bar Element



Figure(2): Components of Nodal Displacements

v_1, u_1, v_2, u_2 – Components of nodal displacements q_1, q_2 in the global co-ordinate system (x,y directions)

x,y – Global co-ordinates

m,n – Local co-ordinates

q_1, q_2 – Displacement at nodes 1,2 in the local co-ordinate system (m,n directions)

$$\{q\} = \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$

$$q_1 = u_1 \cos\theta + v_1 \sin\theta$$

$$q_2 = u_2 \cos\theta + v_2 \sin\theta$$

$$\{q\} = \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} \qquad \begin{aligned} q_1 &= u_1 \cos\theta + v_1 \sin\theta \\ q_2 &= u_2 \cos\theta + v_2 \sin\theta \end{aligned}$$

$$\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}$$

Let $C = \cos\theta$, $S = \sin\theta$

$$\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{bmatrix} C & S & 0 & 0 \\ 0 & 0 & C & S \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}$$

$$\{q\} = \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{bmatrix} C & S & 0 & 0 \\ 0 & 0 & C & S \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}$$

$$\{q\} = [L]\ \{\delta'\}$$

Where,

$$[L] = \begin{bmatrix} C & S & 0 & 0 \\ 0 & 0 & C & S \end{bmatrix}$$

Transformation matrix

Strain Energy

$$U = \frac{1}{2}\{q\}^T [K_l]\{q\}$$

$$U = \frac{1}{2}\{\delta'\}^T [L]^T [K_l][L]\{\delta'\}$$

$$U = \frac{1}{2}\{\delta'\}^T [K]\{\delta'\}$$

Where

$$[K] = [L]^T [K_l] [L] \qquad ...(7)$$

= Element stiffness matrix in global co-ordinate system

Truss is a one dimensional element in the local co-ordinate system. Therefore, element stiffness matrix of a truss element in local co-ordinate system is given by,

$$[K_l] = \frac{AE}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

A – Area of cross-section of truss element

E – Young's modulus

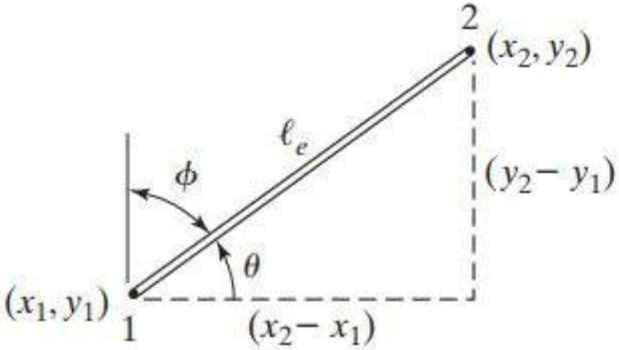
l – Length of truss element

$$[K] = [L]^T [K_l]\,[L]$$

$$[K] = \begin{bmatrix} C & 0 \\ S & 0 \\ 0 & C \\ 0 & S \end{bmatrix} \frac{AE}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} C & S & 0 & 0 \\ 0 & 0 & C & S \end{bmatrix}$$

$$= \frac{AE}{l} \begin{bmatrix} C & 0 \\ S & 0 \\ 0 & C \\ 0 & S \end{bmatrix} \begin{bmatrix} C & S & -C & -S \\ -C & -S & C & S \end{bmatrix}$$

$$[K] = \frac{AE}{l} \begin{bmatrix} C^2 & CS & -C^2 & -CS \\ CS & S^2 & -CS & -S^2 \\ -C^2 & -CS & C^2 & CS \\ -CS & -S^2 & CS & S^2 \end{bmatrix}$$



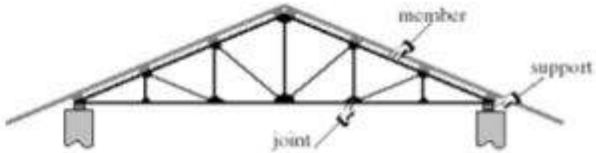
$$\ell = \cos\theta = \frac{x_2 - x_1}{\ell_e}$$

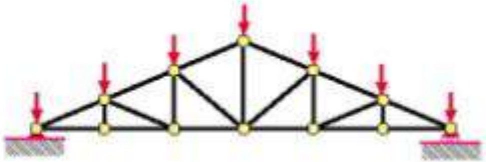
$$m = \cos\phi = \frac{y_2 - y_1}{\ell_e} (= \sin\theta)$$

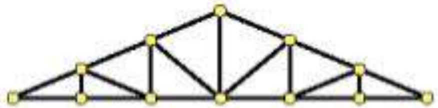
$$\ell_e = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

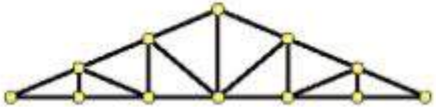
Let $C = \cos\theta$, $S = \sin\theta$

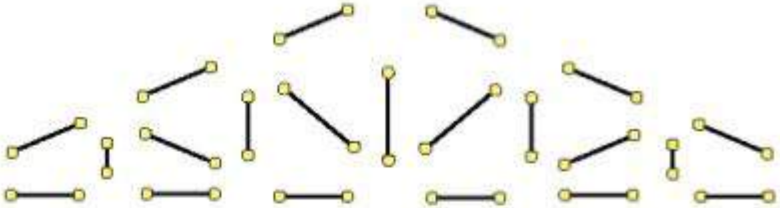
	Bar Element		Truss Element
1.	Displacements of a loaded bar element occurs only in x-direction (horizontal).	1.	The joint displacements of a loaded truss element are neither horizontal nor vertical, but they are resolved into horizontal and vertical components.
2.	The loads in the bar elements are applied in axial direction.	2.	The load applied in the truss elements are either compression or tension.
3.	For joining bar elements, various types of weldings are used.	3.	To join truss elements, pin joints are used.
4.	Stiffness matrix for bar element is given by, $[K] = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$	4.	Stiffness matrix for truss element is given by, $K = \frac{EA}{L}\begin{bmatrix} C^2 & CS & -C^2 & -CS \\ CS & S^2 & -CS & -S^2 \\ -C^2 & -CS & C^2 & CS \\ -CS & -S^2 & CS & S^2 \end{bmatrix}$ Where, $C = \cos\theta$ and $S = \sin\theta$.
5.	Stress in a bar element is, $\sigma = E\varepsilon$ i.e., $\sigma = E[B]\ \{\delta\}$ Where, [B] – Strain displacement matrix $\{\delta\}$ – Nodal displacement.	5.	Stress in a truss element is, $\sigma = \mathrm{E}\varepsilon$ $= \frac{E}{L}\begin{bmatrix} -C & -S & C & S \end{bmatrix} u$ Where, u – Nodal displacement.

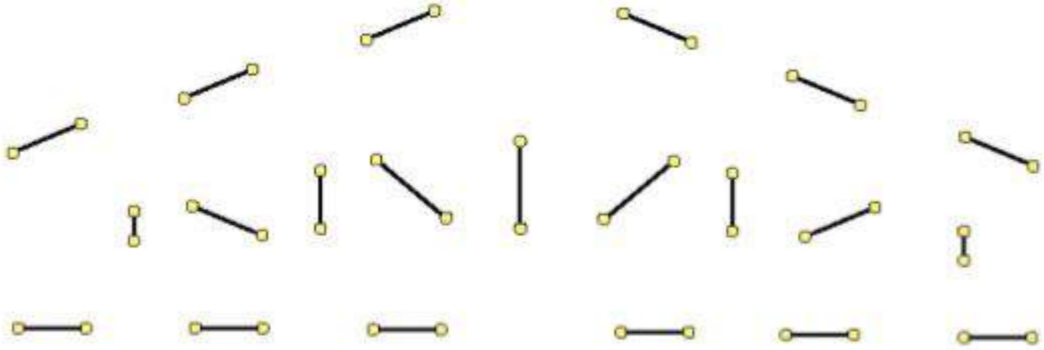

member
support
joint

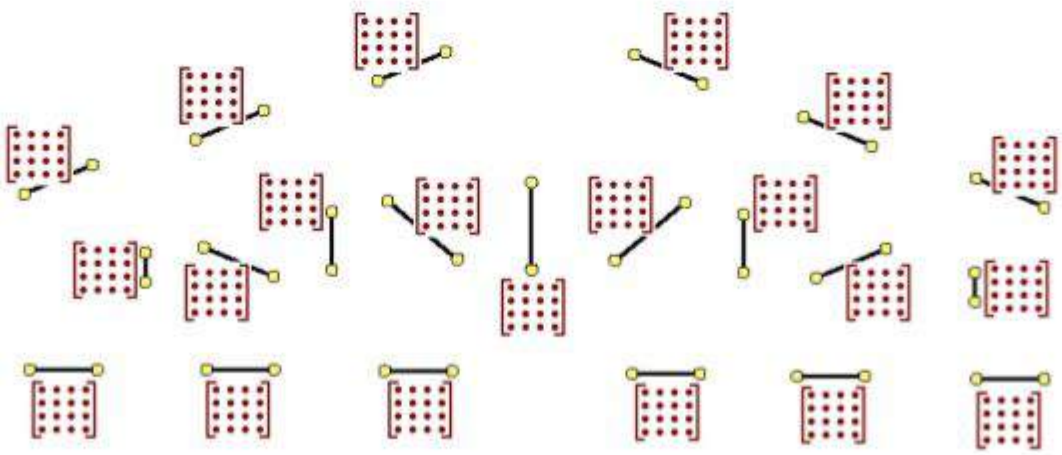


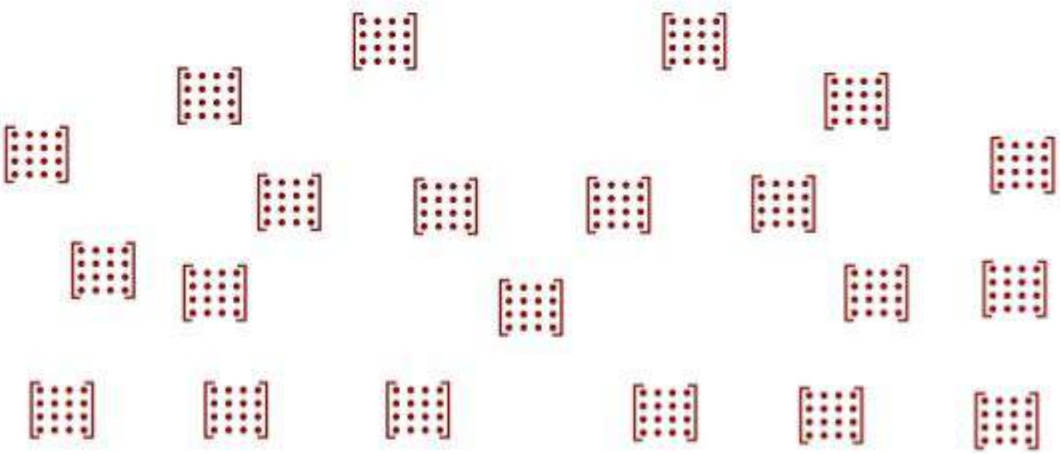


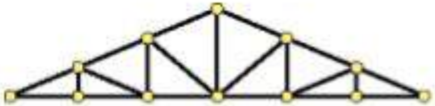


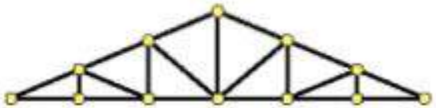


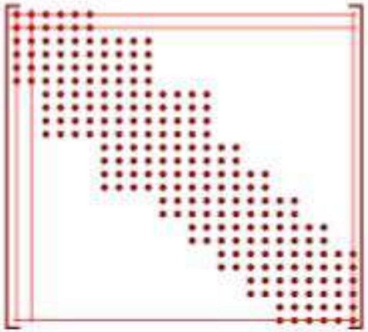


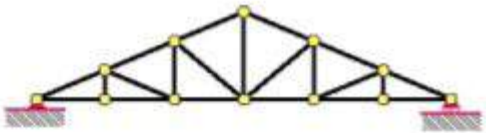


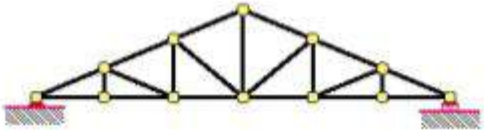


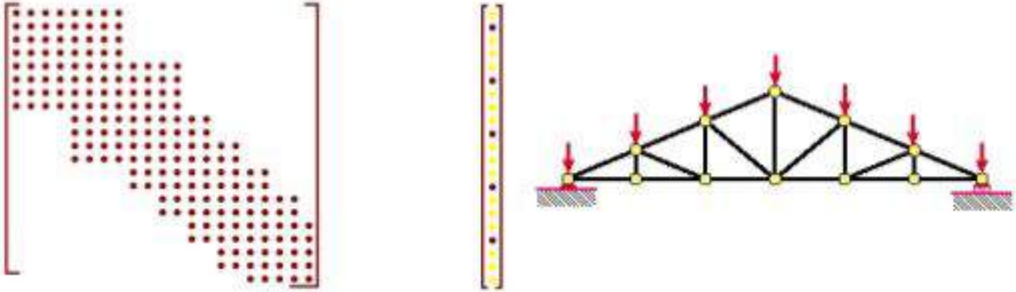


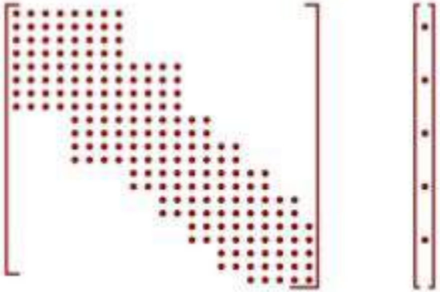


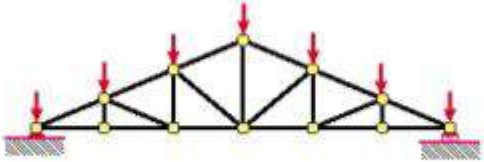


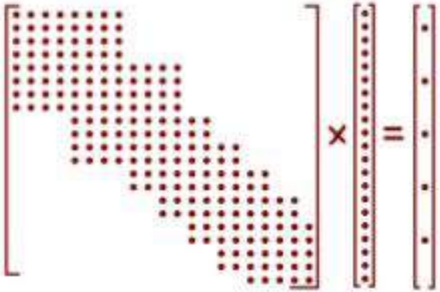


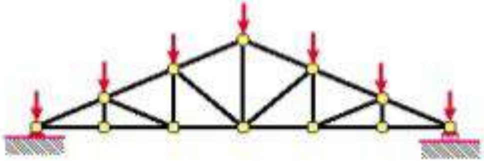








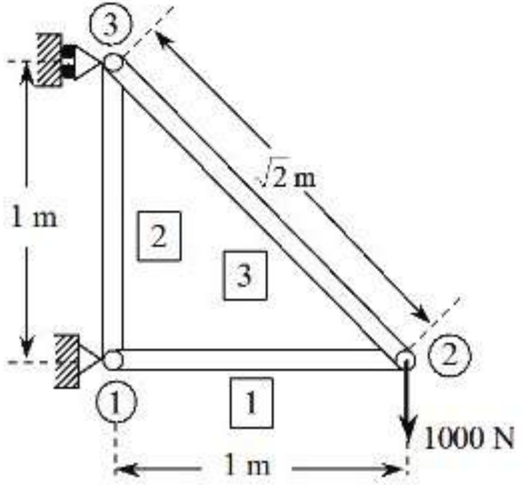




The plane truss shown in figure is composed of members having 0.1 m^2 cross-sectional area and modulus of elasticity E = 70 GPa,

(a) Assemble the global stiffness matrix

(b) Compute the nodal displacements in the global coordinate system.



Figure

Given that,

Area of truss members, $A = 0.1\ \text{m}^2$

Young's modulus, $E = 70\ \text{GPa} = 70 \times 10^9\ \text{N/m}^2$

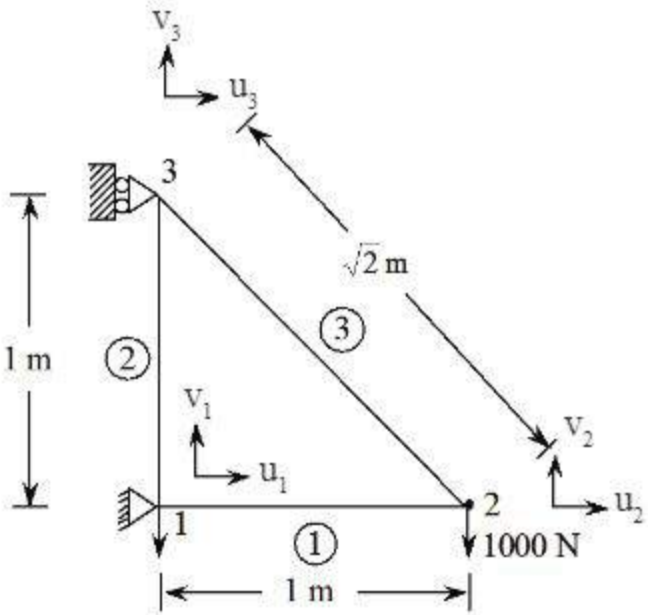


Figure (1): Truss Element

Let u_1, v_1, u_2, v_2, u_3 and v_3 are the nodal displacements in 'x' and 'y' direction respectively.

The stiffness matrix for truss element is given as,

$$[K] = \frac{AE}{L}\begin{bmatrix} C^2 & CS & -C^2 & -CS \\ CS & S^2 & -CS & -S^2 \\ -C^2 & -CS & C^2 & CS \\ -CS & -S^2 & CS & S^2 \end{bmatrix}$$

Where, $C = \cos\theta$ and $S = \sin\theta$

For Element-1

$$\theta_1 = 0^\circ$$

$$C = \cos\theta_1 = \cos 0^\circ = 1$$

$$S = \sin\theta_1 = \sin 0^\circ = 0$$

$$[K_1] = \frac{0.1\times70\times10^9}{1}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= 7\times10^9 \cdot \begin{array}{c} \begin{array}{cccc} u_1 & v_1 & u_2 & v_2 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} u_1 \\ v_1 \\ u_2 \\ v_2 \end{array}$$

For Element-2

$$\theta_2 = 90^\circ \text{ defined from node 1}$$

$$C = \cos\theta_2 = \cos 90^\circ = 0$$

$$S = \sin\theta_2 = \sin 90^\circ = 1$$

$$[K_2] = \frac{0.1\times70\times10^9}{1}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$= 7\times10^9 \begin{array}{c} \begin{array}{cccc} u_1 & v_1 & u_3 & v_3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} u_1 \\ v_1 \\ u_3 \\ v_3 \end{array}$$

For Element-3

$$\sin\theta_3 = \frac{1}{\sqrt{2}}$$

$$\theta_3 = -45^\circ \text{ (cw)}$$

$$= 135^\circ \text{ (ccw)}$$

$\theta_3 = 135^\circ$ defined from node 2 to 3.

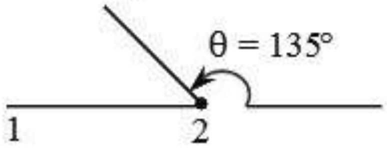


Figure (2)

$$C = \cos\theta_3 = \cos(135^\circ) = \frac{-\sqrt{2}}{2} = -0.707$$

$$S = \sin\theta_3 = \sin(135^\circ) = \frac{\sqrt{2}}{2} = 0.707$$

$$[K_3] = \frac{0.1\times 70\times 10^9}{\sqrt{2}} \times \begin{bmatrix} 0.5 & -0.5 & -0.5 & 0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & -0.5 & 0.5 \end{bmatrix}$$

$$= 7\times 10^9 \begin{array}{c} \begin{array}{cccc} u_2 & v_2 & u_3 & v_3 \end{array} \\ \begin{bmatrix} 0.354 & -0.354 & -0.354 & 0.354 \\ -0.354 & 0.354 & 0.354 & -0.354 \\ -0.354 & 0.354 & 0.354 & -0.354 \\ 0.354 & -0.354 & -0.354 & 0.354 \end{bmatrix} \end{array} \begin{array}{c} \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{array}$$

The global stiffness matrix is calculated as,

$$[K] = [K_1] + [K_2] + [K_3]$$

$$\therefore \quad [K] = 7 \times 10^9 \begin{array}{c} \begin{array}{cccccc} u_1 & v_1 & u_2 & v_2 & u_3 & v_3 \end{array} \\ \left[\begin{array}{cccccc} \; & \; & \; & \; & \; & \; \\ \; & \; & \; & \; & \; & \; \\ \; & \; & \; & \; & \; & \; \\ \; & \; & \; & \; & \; & \; \\ \; & \; & \; & \; & \; & \; \\ \; & \; & \; & \; & \; & \; \end{array} \right] \end{array} \begin{array}{c} \\ u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{array}$$

$$[K] = 7 \times 10^9 \begin{array}{c} \begin{array}{cccccc} u_1 & v_1 & u_2 & v_2 & u_3 & v_3 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ -1 & 0 & 1.354 & -0.354 & -0.354 & 0.354 \\ 0 & 0 & -0.354 & 0.354 & 0.354 & -0.354 \\ 0 & 0 & -0.354 & 0.354 & 0.354 & -0.354 \\ 0 & -1 & 0.354 & -0.354 & -0.354 & 1.354 \end{bmatrix} \end{array} \begin{array}{c} \\ u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{array}$$

Global displacement vector, Global force vector,

$$\{\delta'\} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} \qquad \{F\} = \begin{Bmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ F_{2y} \\ F_{3x} \\ F_{3y} \end{Bmatrix}$$

Nodal Displacements: The finite element matrix equation for truss structure can be written as,

$$[K]\ \{\delta'\} = \{F\}$$

$$7 \times 10^9 \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ -1 & 0 & 1.354 & -0.354 & -0.354 & 0.354 \\ 0 & 0 & -0.354 & 0.354 & 0.354 & -0.354 \\ 0 & 0 & -0.354 & 0.354 & 0.354 & -0.354 \\ 0 & -1 & 0.354 & -0.354 & -0.354 & 1.354 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ F_{2y} \\ F_{3x} \\ F_{3y} \end{Bmatrix}$$

$$7 \times 10^9 \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ -1 & 0 & 1.354 & -0.354 & -0.354 & 0.354 \\ 0 & 0 & -0.354 & 0.354 & 0.354 & -0.354 \\ 0 & 0 & -0.354 & 0.354 & 0.354 & -0.354 \\ 0 & -1 & 0.354 & -0.354 & -0.354 & 1.354 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ F_{2y} \\ F_{3x} \\ F_{3y} \end{Bmatrix}$$

Applying boundary conditions,

i.e., $u_1 = v_1 = u_3 = 0$

Hence, omit 1st, 2nd, 5th row and columns in finite element equations,

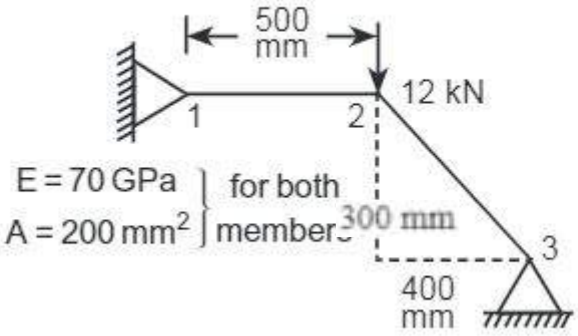
$$7 \times 10^9 \begin{bmatrix} 1.354 & -0.354 & 0.354 \\ -0.354 & 0.354 & -0.354 \\ 0.354 & -0.354 & 1.354 \end{bmatrix} \begin{Bmatrix} u_2 \\ v_2 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -1000 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} u_2 \\ v_2 \\ v_3 \end{Bmatrix} = \begin{bmatrix} 1.354 & -0.354 & 0.354 \\ -0.354 & 0.354 & -0.354 \\ 0.354 & -0.354 & 1.354 \end{bmatrix}^{-1} \begin{Bmatrix} 0 \\ -1000 \\ 0 \end{Bmatrix} \times \frac{1}{7\times10^9}$$

$$\begin{Bmatrix} u_2 \\ v_2 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} -1000 \\ -4824.859 \\ -1000 \end{Bmatrix} \times \frac{1}{7\times10^9}\ \text{m}$$

$$\begin{Bmatrix} u_2 \\ v_2 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} 142.857 \\ 689.266 \\ 142.857 \end{Bmatrix} \times 10^{-6}\ \text{mm}$$

For the two bar truss as shown in figure determine the displacements at node 2 and the stresses in both elements.



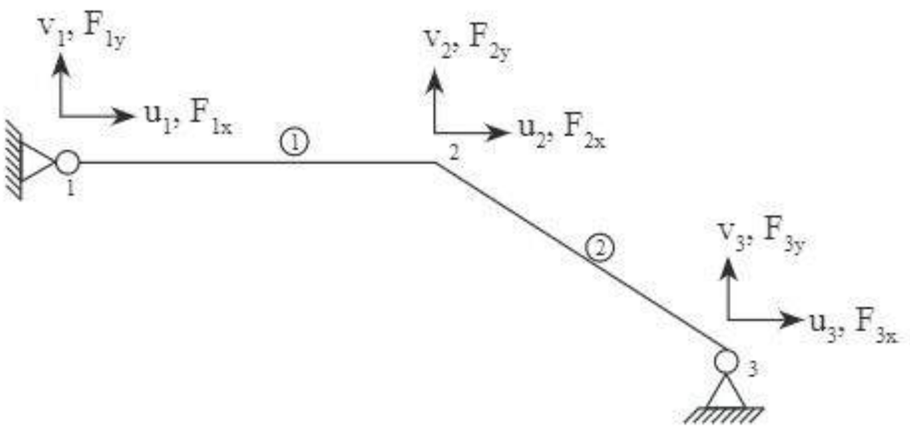
Figure

Given that,

Young's modulus, $E = 70$ GPa $= 70 \times 10^3$ N/mm^2

Area, $A = 200$ mm^2

Truss is divided into two elements as shown in figure.



Figure(1)

Let,

u_1, u_2, u_3 – Displacements along x-axis at nodes 1, 2 and 3 respectively.

v_1, v_2, v_3 – Displacements along y-axis at nodes 1, 2 and 3 respectively.

F_{1x}, F_{2x}, F_{3x} – Forces along x-axis at nodes 1, 2 and 3 respectively.

F_{1y}, F_{2y}, F_{3y} – Forces along x-axis at nodes 1, 2 and 3 respectively.

The stiffness matrix for truss element is given as,

$$[K] = \frac{AE}{L}\begin{bmatrix} C^2 & CS & -C^2 & -CS \\ CS & S^2 & -CS & -S^2 \\ -C^2 & -CS & C^2 & CS \\ -CS & -S^2 & CS & S^2 \end{bmatrix}$$

Where, $C = \cos\theta$ and $S = \sin\theta$

Stiffness matrix for element (1),

$$[K_1] = \frac{A_1 E_1}{L_1}\begin{bmatrix} C^2 & CS & -C^2 & -CS \\ CS & S^2 & -CS & -S^2 \\ -C^2 & -CS & C^2 & CS \\ -CS & -S^2 & CS & S^2 \end{bmatrix}$$

Where,

$C = \cos\theta_1 = \cos 0° = 1$

$(\because$ Element (1) is on x-axis, $\theta_1 = 0)$

$S = \sin\theta_1 = \sin 0° = 0$

Length, $L_1 = 500$ mm

Area, $A_1 = A = 200 \text{ mm}^2$

Young's modulus, $E_1 = E = 70 \times 10^3 \text{ N/mm}^2$

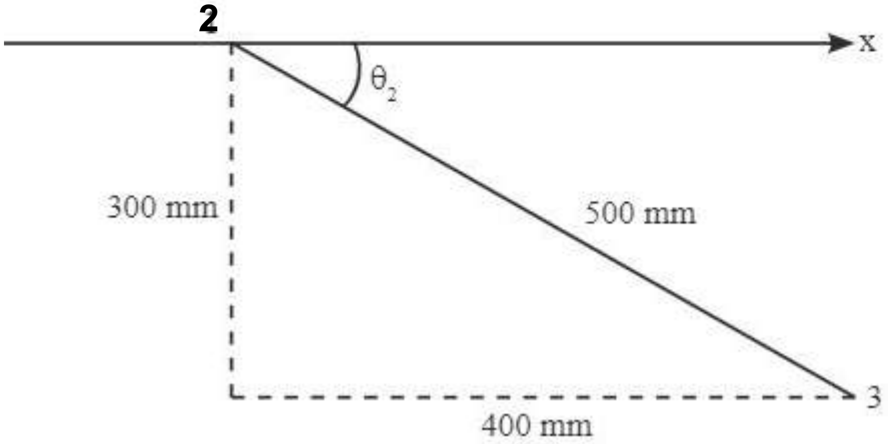
Then,

$$\therefore \quad [K_1] = \frac{200 \times 70 \times 10^3}{500}\begin{array}{c} \begin{array}{cccc} u_1 & v_1 & u_2 & v_2 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}\begin{array}{c} u_1 \\ v_1 \\ u_2 \\ v_2 \end{array}$$

$$= 28 \times 10^3 \begin{array}{c} \begin{array}{cccc} u_1 & v_1 & u_2 & v_2 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}\begin{array}{c} u_1 \\ v_1 \\ u_2 \\ v_2 \end{array}$$

Stiffness matrix for element (2),

$$[K_2] = \frac{A_2 E_2}{L_2}\begin{bmatrix} C^2 & CS & -C^2 & -CS \\ CS & S^2 & -CS & -S^2 \\ -C^2 & -CS & C^2 & CS \\ -CS & -S^2 & CS & S^2 \end{bmatrix}$$



From figure,

$$\sin\theta_2 = \frac{300}{500}$$

$$\therefore \quad \theta_2 = \sin^{-1}(0.6)$$

$$= 36.87°$$

Then,

$$C = \cos\theta_2 = \cos(-36.87) = 0.8$$

('–ve' sign due to clockwise consideration from positive x-axis)

$$S = \sin\theta_2 = \sin(-36.87) = -0.6$$

Length, $l_2 = \sqrt{300^2 + 400^2} = 500$ mm

Area, $A_2 = A = 200$ mm

Young's modulus, $E_2 = E = 70 \times 10^3$ N/mm^2

$$\therefore \quad [K_2] = \frac{200\times 70\times 10^3}{500}\begin{array}{c|c} \begin{array}{cccc} u_2 & v_2 & u_3 & v_3 \end{array} & \\ \begin{bmatrix} 0.8^2 & (0.8\times -0.6) & -0.8^2 & -(0.8\times -0.6) \\ (0.8\times -0.6) & (-0.6)^2 & -(0.8\times -0.6) & -(-0.6)^2 \\ -0.8^2 & -(0.8\times -0.6) & 0.8^2 & (0.8\times -0.6) \\ -(0.8\times -0.6) & -(-0.6)^2 & (0.8\times -0.6) & (-0.6)^2 \end{bmatrix} & \begin{array}{c} u_2 \\ v_2 \\ u_3 \\ v_3 \end{array} \end{array}$$

$$= 28\times 10^3 \begin{array}{cc} \begin{array}{cccc} u_2 & v_2 & u_3 & v_3 \end{array} & \\ \begin{bmatrix} 0.64 & -0.48 & -0.64 & 0.48 \\ -0.48 & 0.36 & 0.48 & -0.36 \\ -0.64 & 0.48 & 0.64 & -0.48 \\ 0.48 & -0.36 & -0.48 & 0.36 \end{bmatrix} & \begin{array}{c} u_2 \\ v_2 \\ u_3 \\ v_3 \end{array} \end{array}$$

Global stiffness matrix, $[K] = [K_1] + [K_2]$

$$\therefore \quad [K] = 28\times 10^3 \begin{array}{cc} \begin{array}{cccccc} u_1 & v_1 & u_2 & v_2 & u_3 & v_3 \end{array} & \\ \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1+0.64 & 0-0.48 & -0.64 & 0.48 \\ 0 & 0 & 0-0.48 & 0+0.36 & 0.48 & -0.36 \\ 0 & 0 & -0.64 & 0.48 & 0.64 & -0.48 \\ 0 & 0 & 0.48 & -0.36 & -0.48 & 0.36 \end{bmatrix} & \begin{array}{c} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{array} \end{array}$$

The finite element equation can be written as,

$$[K]\,[\delta'] = \{F\}$$

$$28 \times 10^3 \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1.64 & -0.48 & -0.64 & 0.48 \\ 0 & 0 & -0.48 & 0.36 & 0.48 & -0.36 \\ 0 & 0 & -0.64 & 0.48 & 0.64 & -0.48 \\ 0 & 0 & 0.48 & -0.36 & -0.48 & 0.36 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ F_{2y} \\ F_{3x} \\ F_{3y} \end{Bmatrix}$$

Nodal Displacements

From nodal boundary conditions,

$$u_1 = 0;\ v_1 = 0;\ u_3 = 0;\ v_3 = 0$$

$$F_{2x} = 0;\ F_{2y} = -12000 \text{ N}$$

Eliminating 1, 2, 5, 6 rows and columns,

$$28 \times 10^3 \begin{bmatrix} 1.64 & -0.48 \\ -0.48 & 0.36 \end{bmatrix} \begin{Bmatrix} u_2 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -12000 \end{Bmatrix}$$

$$1.64 \times 28 \times 10^3 \times u_2 - 0.48 \times 28 \times 10^3 \times v_2 = 0$$

$$-0.48 \times 28 \times 10^3 \times u_2 + 0.36 \times 28 \times 10^3 \times v_2 = -12000$$

Solving equations (2) and (3),

$$u_2 = -0.571 \text{ mm}$$

$$v_2 = -1.95 \text{ mm}$$

$\therefore$ The displacement at node 2 along 'X' and 'Y' directions (i..e, u_2 and v_2) are – 0.571 mm and – 1.95 mm respectively.

Stress Induced in Element (2)]

$$\sigma_2 = \frac{E}{L_2}[-C \quad -S \quad C \quad S]\begin{Bmatrix} u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

$$= \frac{70 \times 10^3}{500}[-0.8 \quad 0.6 \quad 0.8 \quad -0.6]\begin{Bmatrix} -0.571 \\ -1.95 \\ 0 \\ 0 \end{Bmatrix}$$

$$= 140[0.4568 - 1.17 + 0 - 0]$$

$$\therefore \quad \sigma_2 = -99.848 \text{ N/mm}^2$$

$\therefore$ Stress at element '1 - 3' is – 99.848 N/mm^2.

4.3. For the pin-jointed configuration shown in Fig. P4.3, determine the stiffness values K_{11}, K_{12}, and K_{22} of the global stiffness matrix.

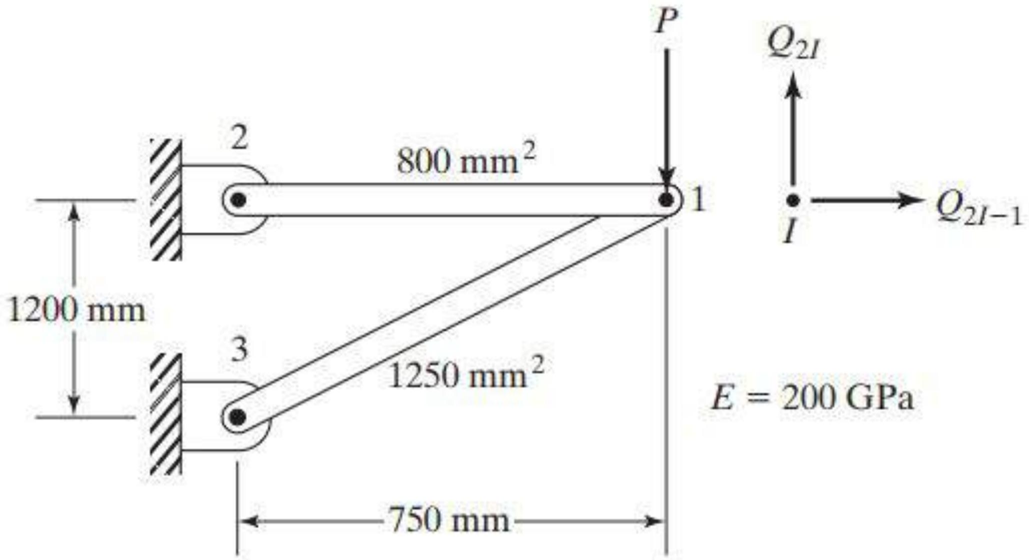
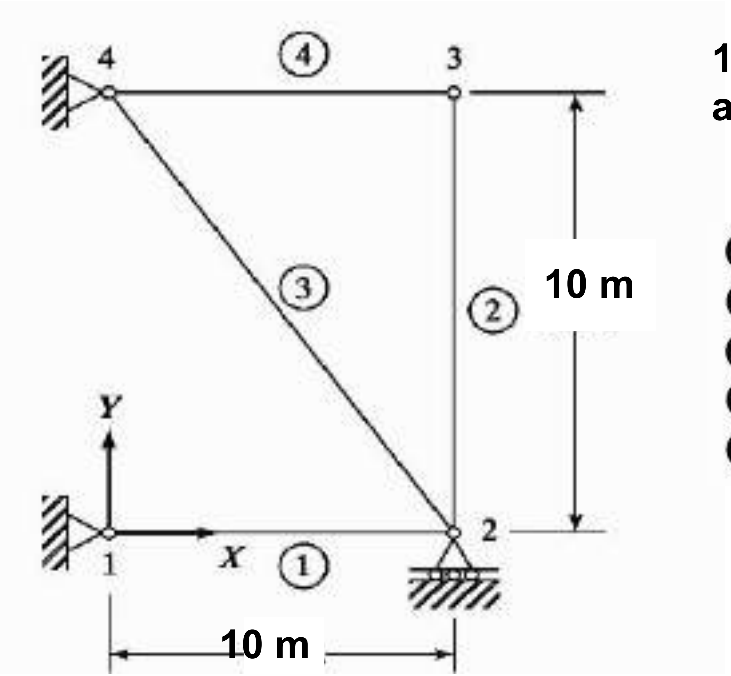


FIGURE P4.3

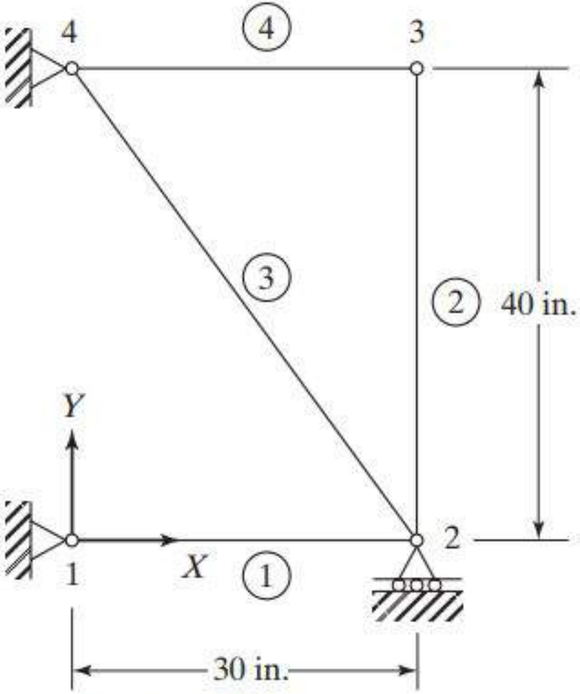


E=210 GPa
A= 100 mm^2

1. For the truss shown in the figure, a horizontal load P (N) is applied in the x direction at node 2.

(a) Write down the element stiffness matrix **k** for each element.
(b) Assemble the **K** matrix.
(c) Using the elimination approach, solve for **Q**.
(d) Evaluate the stress in elements 2 and 3.
(e) Determine the reaction force at node 2 in the *y* direction.

4.4. For the truss in Fig. P4.4, a horizontal load of $P = 2500$ lb is applied in the x direction at node 2.



$E = 30 \times 10^6$ psi
$A = 1.5$ in.^{2} for each member

FIGURE P4.4

(a) Write down the element stiffness matrix **k** for each element.

(b) Assemble the **K** matrix.

(c) Using the elimination approach, solve for **Q**.

(d) Evaluate the stress in elements 2 and 3.

(e) Determine the reaction force at node 2 in the *y* direction.

$$\mathbf{k}^{(1)} = \frac{30\times10^{6}\times1.5}{30}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$E = 30\times10^{6}\ psi$$

$$A = 1.5\ in^{2}$$

$$\mathbf{k}^{(2)} = \frac{30\times10^{6}\times1.5}{40}\begin{array}{c} \begin{array}{cccc} 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \end{array}$$

$$\mathbf{k}^{(3)} = \frac{30\times 10^{6}\times 1.5}{50}\begin{array}{c} \begin{array}{cccc} \quad 7 & \quad\quad 8 & \quad\quad 3 & \quad\quad 4 \end{array} \\ \begin{bmatrix} .36 & -.48 & -.36 & .48 \\ -.48 & .64 & .48 & -.64 \\ -.36 & .48 & .36 & -.48 \\ .48 & -.64 & -.48 & .64 \end{bmatrix} \end{array}$$

$$\mathbf{k}^{(4)} = \frac{30\times 10^{6}\times 1.5}{30}\begin{array}{c} \begin{array}{cccc} 7 & \quad 8 & \quad 5 & \quad 6 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$
\mathbf{K} = \frac{30\times10^{6}\times1.5}{600}
\begin{array}{c}
\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\
\begin{bmatrix}
20 & 0 & -20 & 0 & 0 & 0 & 0 & 0 \\
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & 24.32 & -5.76 & 0 & 0 & -4.32 & 5.76 \\
 & & & 22.68 & 0 & -15 & 0 & -7.68 \\
 & & & & 20 & 0 & -20 & 0 \\
 & & & & & 15 & 0 & 0 \\
 & Sym & & & & & 24.32 & -5.76 \\
 & & & & & & & 7.68
\end{bmatrix}
\end{array}
$$

Solution of $\mathbf{K\,Q} = \mathbf{F}$

$$
\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \\ Q_7 \\ Q_8 \end{Bmatrix}
$$

(c) Eliminating dof's 1,2,4,7,8,

K Q = F is

$$\frac{30\times10^6\times1.5}{600}\begin{array}{c}\begin{array}{ccc}3 & 5 & 6\end{array}\\ \begin{bmatrix}24.32 & 0 & 0\\ 0 & 20 & 0\\ 0 & 0 & 15\end{bmatrix}\end{array}\begin{Bmatrix}Q_3\\ Q_5\\ Q_6\end{Bmatrix}=\begin{Bmatrix}4000\\ 0\\ 0\end{Bmatrix}$$

The solution is

$$Q_3 = 219.3\times10^{-5}\,in$$
$$Q_5 = 0$$
$$Q_6 = 0$$

(d)

$$\sigma = \frac{E}{l_e}[-\ell \quad -m \quad \ell \quad m]\mathbf{q}$$

$$\sigma_2 = \frac{30\times10^6}{40}[0 \quad -1 \quad 0 \quad 1]\begin{Bmatrix} 219.3\times10^{-5} \\ 0 \\ 0 \\ 0 \end{Bmatrix} = 0$$

(Check with free body of Node 3)

$$\sigma_3 = \frac{30\times10^6}{50}[-.6 \quad .8 \quad +.6 \quad -.8]\begin{Bmatrix} 0 \\ 0 \\ 219.3\times10^{-5} \\ 0 \end{Bmatrix} = 789.5\,psi$$

$$
\mathbf{K} = \frac{30\times10^{6}\times1.5}{600}
\begin{array}{c}
\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{matrix} \\
\begin{bmatrix}
20 & 0 & -20 & 0 & 0 & 0 & 0 & 0 \\
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & 24.32 & -5.76 & 0 & 0 & -4.32 & 5.76 \\
 & & & 22.68 & 0 & -15 & 0 & -7.68 \\
 & & & & 20 & 0 & -20 & 0 \\
 & & & & & 15 & 0 & 0 \\
 & Sym & & & & & 24.32 & -5.76 \\
 & & & & & & & 7.68
\end{bmatrix}
\end{array}
$$

e) Reaction of Node 2 in y- direction is

$$R_4 = \sum_{j=1}^{8} K_{4j}\, Q_j$$

$$= \frac{30\times10^{6}\times1.5}{600}\,[-5.76\times219.3\times10^{-5}] = -947.4lb \quad (downward\ \ pull)$$

Unit 2
Beams

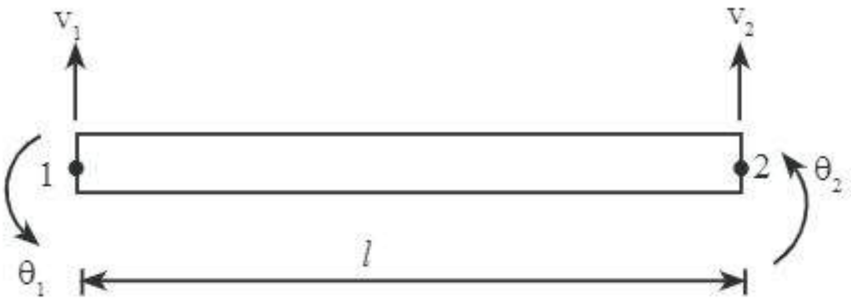
	Bar Element		Beam Element
1.	Bar is a structural element that is subjected to only axial loading.	1.	Beam is a structural element that is subjected to transverse loading
2.	When a bar element is loaded, it is described by the **--axial--** displacements only.	2.	When a beam element is loaded, it is described by the transverse displacements and rotational (slope) displacements.
3.	Stiffness matrix for a bar element is given by, $[K] = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$	3.	Stiffness matrix for a beam element is given by, $[K] = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$
4.	Bar elements are used in model cables, prismatic structural members and ropes.	4.	Long horizontal members used in buildings, bridges and shafts are some of the examples of beams.

Assumptions used in beam elements are:

1. The beam elements are straight and prismatic.
2. The material of beam is linearly-elastic, isotropic and homogeneous.
3. The cross-section of beam is either constant or varies smoothly. Deformation of cross-section does not occur in its plane, but subjected to warping in longitudinal direction.
4. The transverse shear and axial force effects are assumed to be negligible. In case of bending moment deformation, internal strain energy of the beam element is considered.
5. The resultant of stresses (i.e.,internal moments) are determined by Euler-Bernoulli theories for bending stress and Timoshenko theory for torsional stress.
6. The beam elements have larger displacements and smaller strains.
7. External load applied on a beam is static and conservative.

Derivation of shape functions for beam element

$x = 0, v = v_1, \dfrac{dv}{dx} = \theta_1$



$x = l, v = v_2, \dfrac{dv}{dx} = \theta_2$

Figure: Beam Element

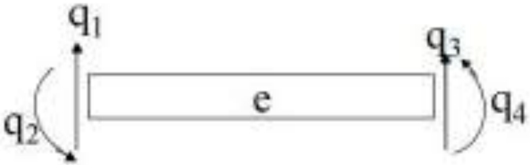
The above figure shows a beam of length 'l' upon loading the beam will have four displacements. v_1 v_2 are the Transverse displacements and θ_1, θ_2 are rotational displacement.

The polynomial function of a beam element of two nodes and with four displacements is given by,

$$v(x) = a_1 + a_2x + a_3x^2 + a_4x^3 \quad ...(1)$$

v – Transverse displacement

a_1, a_2, a_3, a_4 – polynomial coefficients



$$q = [q_1, q_2, q_3, q_4]^T$$
$$= [v_1, v_1', v_2, v_2']$$

$$v(x) = a_1 + a_2x + a_3x^2 + a_4x^3 \quad ...(1)$$

Upon derivating equation (1),

$$\frac{dv}{dx} = a_2 + 2a_3x + 3a_4x^2 \quad ...(2)$$

Applying boundary conditions to equation (2),

At

$$x = 0, v = v_1, \frac{dv}{dx} = \theta_1$$

At

$$x = l, v = v_2, \frac{dv}{dx} = \theta_2$$

From equation (1) and (2),

$$v_1 = a_1 \quad ...(3)$$

$$\theta_1 = a_2 \quad ...(4)$$

$$v_2 = a_1 + a_2l + a_3l^2 + a_4l^3 \quad ...(5)$$

$$\theta_2 = a_2 + 2a_3l + 3a_4l^2 \quad ... (6)$$

From equation (5), (4), (3)

$$v_2 = a_1 + a_2 l + a_3 l^2 + a_4 l^3$$

$$v_2 = v_1 + \theta_1 l + a_3 l^2 + a_4 l^3$$

$$a_3 l^2 + a_4 l^3 = v_2 - v_1 - \theta_1 l \qquad \text{...(7)}$$

From equation (6), (4), (3)

$$\theta_2 = a_2 + 2a_3 l + 3a_4 l^2$$

$$\theta_2 = \theta_1 + 2a_3 l + 3a_4 l^2$$

$$2a_3 l + 3a_4 l^2 = \theta_2 - \theta_1 \qquad \text{...(8)}$$

Multiplay equation (7) by '3' and equation (8) by 'l'

$$3a_3 l^2 + 3a_4 l^3 = 3v_2 - 3v_1 - 3\theta_1 l$$

$$2a_3 l^2 + 3a_4 l^3 = \theta_2 l - \theta_1 l$$

$$\ominus \quad \ominus \quad \ominus \quad \oplus$$

$$a_3 l^2 = 3v_2 - 3v_1 - 3\theta_1 l - \theta_2 l + \theta_1 l$$

$$a_3 l^2 = 3v_2 - 3v_1 - 2\theta_1 l - \theta_2 l$$

$$a_3 = \frac{3}{l^2}(v_2 - v_1) - \frac{1}{l}(2\theta_1 + \theta_2)$$

Substitute the 'a_3' value in equation (7),

$$a_4 l^3 = v_2 - v_1 - \theta_1 l - a_3 l^2$$

$$a_4 l^3 = v_2 - v_1 - \theta_1 l - \left[\frac{3}{l^2}(v_2 - v_1) - \frac{1}{l}(2\theta_1 + \theta_2)\right] l^2$$

$$= v_1 - v_1 - \theta_1 l - 3(v_2 - v_1) + l(2\theta_1 + \theta_2)$$

$$= v_2 - v_1 - \theta_1 l - 3v_2 + 3v_1 + 2l\theta_1 + \theta_2 l$$

$$a_4 l^3 = -2v_2 + 2v_1 + \theta_1 l + \theta_2 l$$

$$a_4 = -\frac{2}{l^3}v_2 + \frac{2}{l^3}v_1 + \frac{\theta_1}{l^2} + \frac{\theta_2}{l^2}$$

$$v(x) = a_1 + a_2 x + a_3 x^2 + a_4 x^3$$

$$v(x) = v_1 + \theta_1 x + \left[\frac{3}{l^3}(v_2 - v_1) - \frac{1}{l}(2\theta_1 + \theta_2)\right]x^2 + \left[\frac{-2}{l^3}v_2 + \frac{2}{l^3}v_1 + \frac{\theta_1}{l^2} + \frac{\theta_2}{l^2}\right]x^3$$

$$v(x) = v_1 + \theta_1 x + \frac{3}{l^2}v_2 x^2 - \frac{3}{l^2}v_1 x^2 - \frac{1}{l}2\theta_1 x^2 - \frac{1}{l}\theta_2 x^2 - \frac{2}{l^3}v_2 x^3 + \frac{2}{l^3}v_1 x^3 + \frac{\theta_1}{l^2}.x^3 + \frac{\theta_2}{l^2}.x^3$$

$$= v_1\left[1 - \frac{3}{l^2}x^2 + \frac{2}{l^3}x^3\right] + \theta_2\left[x - \frac{2}{l}x^2 + \frac{x^3}{l^2}\right] + v_2\left[\frac{3}{l^2}x^2 - \frac{2x^3}{l^3}\right] + \theta_2\left[-\frac{x^2}{l} + \frac{x^3}{l^2}\right]$$

$$v = N_1 v_1 + N_2\theta_1 + N_3 v_2 + N_4\theta_2 \text{ (or) } N_1\delta_1 + N_2\delta_2 + N_3\delta_3 + N_4\delta_4$$

N_1, N_2, N_3, N_4 – Shape functions of a beam element

$v_1, \theta_1\ v_2, \theta_2$ – Nodal displacements of a beam element

$$v = N_1 v_1 + N_2\theta_1 + N_3 v_2 + N_4\theta_2 \text{ (or) } N_1\delta_1 + N_2\delta_2 + N_3\delta_3 + N_4\delta_4$$

N_1, N_2, N_3, N_4 – Shape functions of a beam element

$v_1, \theta_1\ v_2, \theta_2$ – Nodal displacements of a beam element

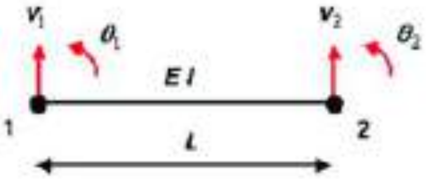
Where,

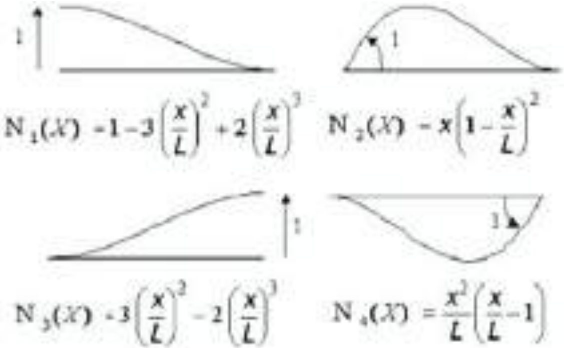
$$N_1 = 1 - \frac{3x^3}{l^2} + \frac{2x^2}{l^3}$$

$$N_2 = x - \frac{2x^2}{l} + \frac{x^3}{l^2}$$

$$N_3 = \frac{3x^2}{l^2} - \frac{2x^3}{l^3}$$

$$N_4 = \frac{-x^2}{l} + \frac{x^3}{l^2}$$





Element Stiffness Matrix (K)_4by4

Strain energy in an element of length dx is

$$\sigma = -\frac{M}{I}y$$

$$\epsilon = \sigma / E$$

$$dU = \frac{1}{2}\int_A \sigma\varepsilon dA dx$$

$$= \frac{1}{2}\left(\frac{M^2}{EI^2}\int_A y^2 dA\right)dx$$

$\int_A y^2 dA$ *is the moment of inertia I*

The total strain energy for the beam is given by-

$$d^2v / dx^2 = M / EI$$

$$U = \frac{1}{2}\int_0^L EI\left(d^2v / dx^2\right)dx$$

v – Beam deflection

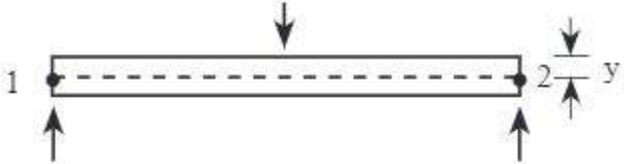


Figure: Beam Element

$$U = \frac{EI}{2}\int_0^l \left[\frac{d^2v}{dx^2}\right]^2 . dx$$

From nodal displacement equation,

$$v = N_1 v_1 + N_2\theta_1 + N_3 v_2 + N_4\theta_4$$

(or)

$$v = N_1\delta_1 + N_2\delta_2 + N_3\delta_3 + N_4\delta_4$$

Differentiating twice on both sides,

$$\frac{d^2v}{dx^2} = \frac{d^2N_1}{dx^2}\delta_1 + \frac{d^2N_2}{dx^2}\delta_2 + \frac{d^2N_3}{dx^2}\delta_3 + \frac{d^2N_4}{dx^2}\delta_4$$

Let,

$$B_1 = \frac{d^2N_1}{dx^2}$$

$$B_2 = \frac{d^2N_2}{dx^2}$$

$$B_3 = \frac{d^2N_3}{dx^2}$$

$$B_4 = \frac{d^2N_4}{dx^2}$$

Where,

$$N_1 = 1 - \frac{3x^3}{l^2} + \frac{2x^2}{l^3}$$

$$N_2 = x - \frac{2x^2}{l} + \frac{x^3}{l^2}$$

$$N_3 = \frac{3x^2}{l^2} - \frac{2x^3}{l^3}$$

$$N_4 = \frac{-x^2}{l} + \frac{x^3}{l^2}$$

$$\frac{d^2v}{dx^2} = B_1\delta_1 + B_2\delta_2 + B_3\delta_3 + B_4\delta_4$$

$$\frac{d^2v}{dx^2} = B_1\delta_1 + B_2\delta_2 + B_3\delta_3 + B_4\delta_4$$

Expressing the above equation in matrix form,

$$\frac{d^2v}{dx^2} = \begin{bmatrix} B_1 & B_2 & B_3 & B_4 \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{Bmatrix}$$

$$\frac{d^2v}{dx^2} = [B]\ \{\delta\}$$

Squaring on both sides,

$$\left[\frac{d^2v}{dx^2}\right]^2 = [[B]\{\delta\}]^2$$

$$\left[\frac{d^2v}{dx^2}\right]^2 = \{\delta\}^T [B]^T [B]\ \{\delta\}$$

$$U = \frac{EI}{2}\int_0^l \{\delta\}^T [B]^T [B]\{\delta\}\, dx$$

$$U = \frac{1}{2}\{\delta\}^T \left[EI \int_0^l [B]^T [B].dx \right] \{\delta\}$$

$$U = \frac{1}{2}\{\delta\}^T [k]\{\delta\}$$

$$U = \frac{1}{2}\{\delta\}^T [k] \{\delta\}$$

Where,

$$[k] = EI \int_0^l [B]^T [B].dx$$

$$[k] = EI \int_0^l \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix} \begin{bmatrix} B_1 & B_2 & B_3 & B_4 \end{bmatrix} dx$$

$$[k] = EI \int_0^l \begin{bmatrix} B_1^2 & B_1B_2 & B_1B_3 & B_1B_4 \\ B_1B_2 & B_2^2 & B_2B_3 & B_2B_4 \\ B_1B_3 & B_2B_3 & B_3^2 & B_3B_4 \\ B_1B_4 & B_2B_4 & B_3B_4 & B_4^2 \end{bmatrix} dx$$

From shape functions,

$$B_1 = \frac{d^2 N_1}{dx^2}$$

$$N_1 = 1 - \frac{3x^2}{l^2} + \frac{2x^3}{l^3}$$

$$\frac{dN_1}{dx} = \frac{-6x}{l^2} + \frac{6x^2}{l^3}$$

$$\frac{d^2 N}{dx^2} = \frac{-6}{l^2} + \frac{12x}{l^3}$$

$$\therefore B_1 = \frac{-6}{l^2} + \frac{12x}{l^3}$$

Similarly,

For,

$$N_2 = x - \frac{2x^2}{l} + \frac{x^3}{l^2}$$

$$B_2 = \frac{d^2 N_2}{dx^2} = \frac{-4}{l} + \frac{6x}{l^2}$$

Now,

$$\int_0^l B_1^2 .dx = \int_0^l \left(\frac{-6}{l^2} + \frac{12x}{l^3}\right)^2 .dx$$

$$= \int_0^l \left(\frac{36}{l^4} + \frac{144x^2}{l^6} - \frac{144x}{l^5}\right). dx$$

$$= \left(\frac{36x}{l^4} + \frac{144x^3}{3l^6} - \frac{144x^2}{2l^5}\right)_0^l$$

$$= \frac{36l}{l^4} + \frac{144l^3}{3l^6} - \frac{72l^2}{l^5}$$

$$= \frac{36}{l^3} + \frac{48}{l^3} - \frac{72}{l^3}$$

$$\int_0^l B_1^2 .dx = \frac{12}{l^3}$$

Similarly solve for all the values 'B' in equation (3), i.e.,

$B_1 B_2$, $B_1 B_3$, $B_1 B_4$ ....$B_3 B_4$, B_4^2

The end matrix will be in the following form,

$$\begin{bmatrix} B_1^2 & B_1B_2 & B_1B_3 & B_1B_4 \\ B_1B_2 & B_2^2 & B_2B_3 & B_2B_4 \\ B_1B_3 & B_2B_3 & B_3^2 & B_3B_4 \\ B_1B_4 & B_2B_4 & B_3B_4 & B_4^2 \end{bmatrix} = \begin{bmatrix} \frac{12}{l^3} & \frac{6}{l^2} & \frac{-12}{l^3} & \frac{6}{l^2} \\ \frac{6}{l^2} & \frac{4}{l} & \frac{-6}{l^2} & \frac{2}{l} \\ \frac{-12}{l^3} & \frac{-6}{l^2} & \frac{12}{l^3} & \frac{-6}{l^2} \\ \frac{6}{l^2} & \frac{2}{l} & \frac{-6}{l^2} & \frac{4}{l} \end{bmatrix} = \frac{1}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$$

We have

$$[k] = EI \int_0^l \begin{bmatrix} B_1^2 & B_1B_2 & B_1B_3 & B_1B_4 \\ B_1B_2 & B_2^2 & B_2B_3 & B_2B_4 \\ B_1B_3 & B_2B_3 & B_3^2 & B_3B_4 \\ B_1B_4 & B_2B_4 & B_3B_4 & B_4^2 \end{bmatrix} dx$$

$$[k] = \frac{EI}{l^3} \begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$$

Force Vector

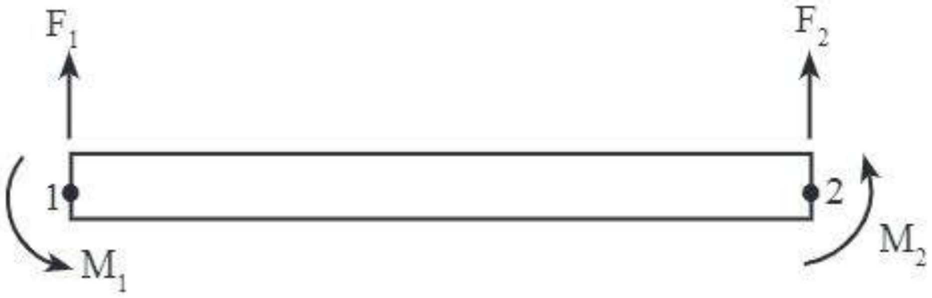


Figure: Beam Element

F_1, F_2 – Shear forces in upward direction at nodes 1,2.

M_1, M_2 – Bending Moments in counterclockwise at node 1,2.

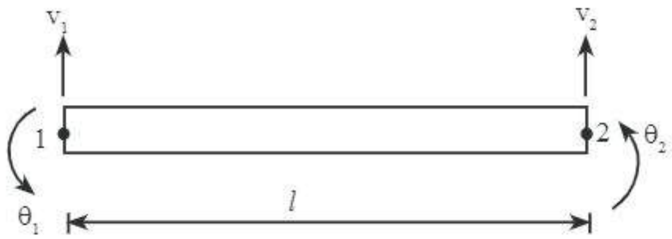


Figure: Beam Element

Force vector for a beam element is given by,

$$\{F\} = [K]\,[\delta]$$

$\{F\}$ – Force vector

$$= \begin{Bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{Bmatrix}$$

$\{\delta\}$ – Nodal displacement vector

$$= \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix}$$

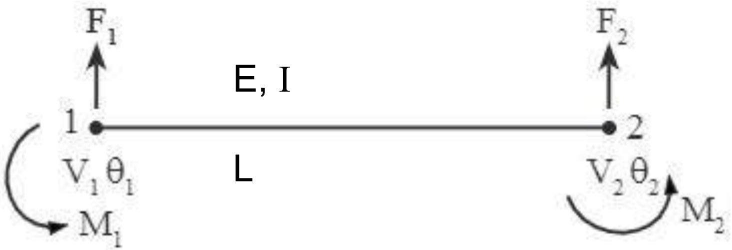


Figure (2): FE Model

$$\begin{Bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{Bmatrix} = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix}$$

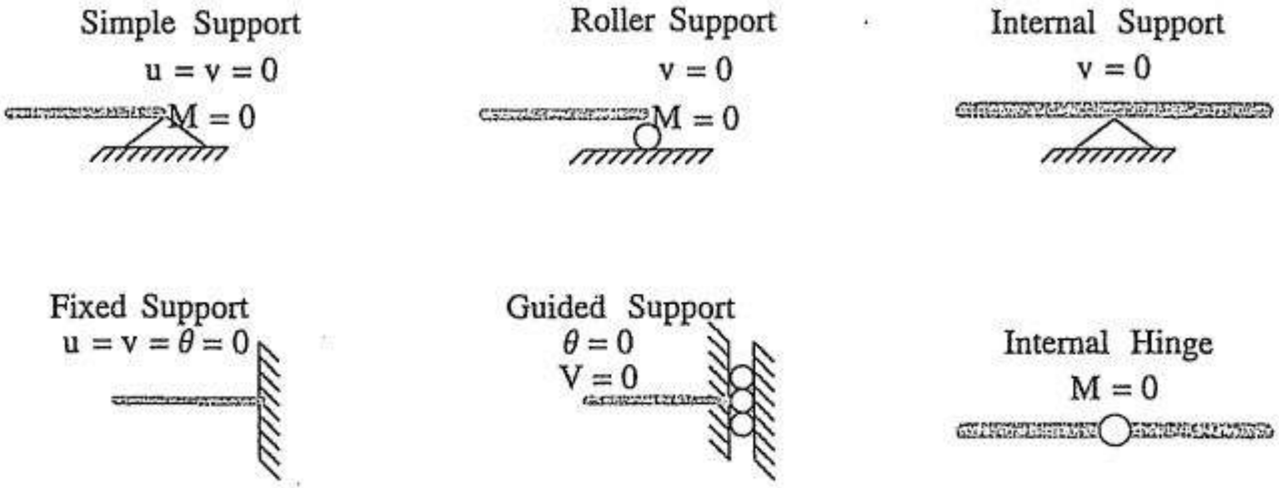
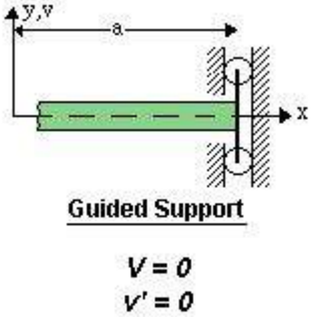
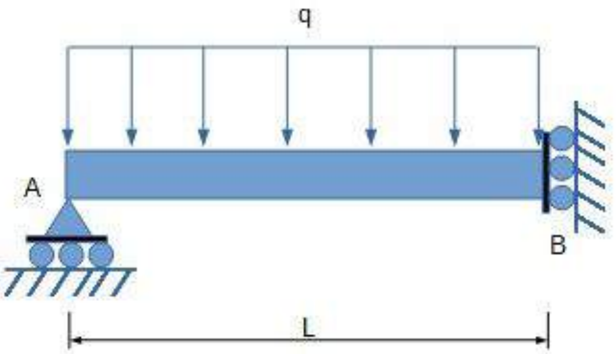


Figure 4.12. Typical beam boundary conditions





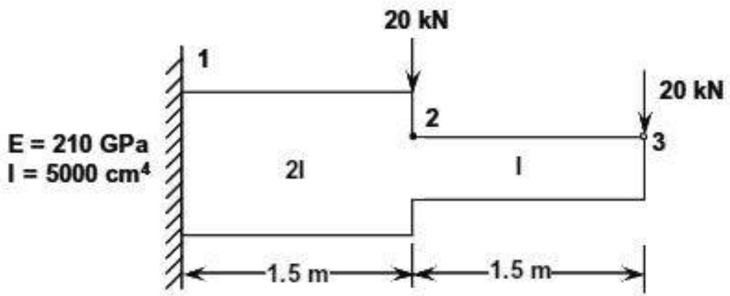
Deflection: v

Rotation: $\theta = \dfrac{\partial v}{\partial x}$

Bending moment: $M = EI\dfrac{\partial^2 v}{\partial x^2}$

Shear force: $V = \dfrac{\partial}{\partial x}\left[EI\dfrac{\partial^2 v}{\partial x^2}\right]$

For the cantilever beam shown in the figure determine the nodal displacements. Construct the shear force and bending moment diagrams. Compare the results. Given E = 210 GPa and I = 5000 cm^4.



Figure

Given that,

Young's modulus, $E = 210\text{ GPa} = 210 \times 10^9\text{ N/m}^2$

Moment of inertia, $I = 5000\text{ cm}^4 = 5000 \times 10^{-8}\text{ m}^4$

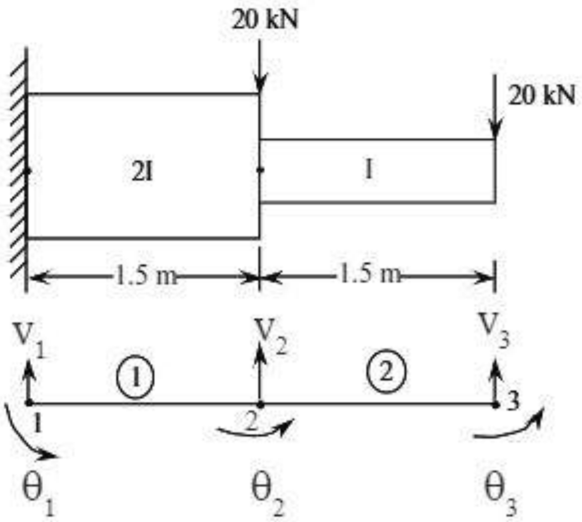


Figure (1): Discretization of Beam

Let v_1, θ_1, v_2, θ_2, v_3, θ_2 are the nodal displacements.

The element stiffness matrix for beam element is given by,

$$[K] = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$$

Element-1

Stiffiness matrix for element (1) is given by,

$$[K_1] = \frac{E_1 I_1}{l_1^3}\begin{bmatrix} 12 & 6l_1 & -12 & 6l_1 \\ 6l_1 & 4l^2 & -6l_1 & 2l^2 \\ -12 & -6l_1 & 12 & -6l_1 \\ 6l_1 & 2l_1^2 & -6l_1 & 4l_1^2 \end{bmatrix}$$

Where, $I_1 = 2I$

$$[K_1] = \frac{210 \times 10^9 \times 5000 \times 10^{-8} \times 2}{(1.5)^3} \begin{bmatrix} 12 & 6(1.5) & -12 & 6(1.5) \\ 6(1.5) & 4(1.5)^2 & -6(1.5) & 2(1.5)^2 \\ -12 & -6(1.5) & 12 & -6(1.5) \\ 6(1.5) & 2(1.5)^2 & -6(1.5) & 4(1.5)^2 \end{bmatrix}$$

$$= 6.22 \times 10^6 \cdot \begin{bmatrix} 12 & 9 & -12 & 9 \\ 9 & 9 & -9 & 4.5 \\ -12 & -9 & 12 & -9 \\ 9 & 4.5 & -9 & 9 \end{bmatrix}$$

$$= 0.622 \times 10^7 \cdot \begin{bmatrix} 12 & 9 & -12 & 9 \\ 9 & 9 & -9 & 4.5 \\ -12 & -9 & 12 & -9 \\ 9 & 4.5 & -9 & 9 \end{bmatrix}$$

$$[K_1] = 10^7 \cdot \begin{array}{c} \begin{array}{cccc} v_1 & \theta_1 & v_2 & \theta_2 \end{array} \\ \begin{bmatrix} 7.464 & 5.598 & -7.464 & 5.598 \\ 5.598 & 5.598 & -5.598 & 2.799 \\ -7.464 & -5.598 & 7.464 & -5.598 \\ 5.598 & 2.799 & -5.598 & 5.598 \end{bmatrix} \end{array} \begin{array}{c} \\ v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{array}$$

Element-2

Stiffness matrix for element (2) is given by,

$$[K_2] = \frac{E_2 I_2}{l_2^3}\begin{bmatrix} 12 & 6l_2 & -12 & 6l_2 \\ 6l_2 & 4l_2^2 & -6l_2 & 2l_2^2 \\ -12 & -6l_2 & 12 & -6l_2 \\ 6l_2 & 2l_2^2 & 6l_2 & 4l_2^2 \end{bmatrix}$$

Where, $I_2 = I$

$$[K_2] = \frac{210\times10^9\times5000\times10^{-8}}{(1.5)^3}\begin{bmatrix} 12 & 6(1.5) & -12 & 6(1.5) \\ 6(1.5) & 4(1.5)^2 & -6(1.5) & 2(1.5)^2 \\ -12 & -6(1.5) & 12 & -6(1.5) \\ 6(1.5) & 2(1.5)^2 & -6(1.5) & 4(1.5)^2 \end{bmatrix} = 0.311\times10^7\begin{bmatrix} 12 & 9 & -12 & 9 \\ 9 & 9 & -9 & 4.5 \\ -12 & -9 & 12 & -9 \\ 9 & 4.5 & -9 & 9 \end{bmatrix}$$

$$[K_2] = 10^7 \cdot \begin{array}{c} \begin{array}{cccc} v_2 & \theta_2 & v_3 & \theta_3 \end{array} \\ \begin{bmatrix} 3.732 & 2.799 & -3.732 & 2.799 \\ 2.799 & 2.799 & -2.799 & 1.4 \\ -3.732 & -2.799 & 3.732 & -2.799 \\ 2.799 & 1.4 & -2.799 & 2.799 \end{bmatrix} \end{array} \begin{array}{c} v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{array}$$

$$[K_1] = 10^7 . \begin{array}{c} \begin{array}{cccc} v_1 & \theta_1 & v_2 & \theta_2 \end{array} \\ \begin{bmatrix} 7.464 & 5.598 & -7.464 & 5.598 \\ 5.598 & 5.598 & -5.598 & 2.799 \\ -7.464 & -5.598 & 7.464 & -5.598 \\ 5.598 & 2.799 & -5.598 & 5.598 \end{bmatrix} \end{array} \begin{array}{c} \\ v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{array}$$

$$[K_2] = 10^7 . \begin{array}{c} \begin{array}{cccc} v_2 & \theta_2 & v_3 & \theta_3 \end{array} \\ \begin{bmatrix} 3.732 & 2.799 & -3.732 & 2.799 \\ 2.799 & 2.799 & -2.799 & 1.4 \\ -3.732 & -2.799 & 3.732 & -2.799 \\ 2.799 & 1.4 & -2.799 & 2.799 \end{bmatrix} \end{array} \begin{array}{c} \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{array}$$

Global stiffness matrix,

$$[K] = [K_1] + [K_2]$$

$$[K] = 10^7 \begin{array}{c} \begin{array}{cccccc} v_1 & \theta_1 & v_2 & \theta_2 & v_3 & \theta_3 \end{array} \\ \begin{bmatrix} 7.464 & 5.598 & -7.464 & 5.598 & 0 & 0 \\ 5.598 & 5.598 & -5.598 & 2.799 & 0 & 0 \\ -7.464 & -5.598 & 11.196 & -2.799 & -3.732 & 2.799 \\ 5.598 & 2.799 & -2.799 & 8.397 & -2.799 & 1.4 \\ 0 & 0 & -3.732 & -2.799 & 3.732 & -2.799 \\ 0 & 0 & 2.799 & 1.4 & -2.799 & 2.799 \end{bmatrix} \end{array} \begin{array}{c} \\ v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{array}$$

The finite element equation is given by,

$$[K]\{\delta\}=\{F\}$$

$$10^7\begin{bmatrix} 7.464 & 5.598 & -7.464 & 5.598 & 0 & 0 \\ 5.598 & 5.598 & -5.598 & 2.799 & 0 & 0 \\ -7.464 & -5.598 & 11.196 & -2.799 & -3.732 & 2.799 \\ 5.598 & 2.799 & -2.799 & 8.397 & -2.799 & 1.4 \\ 0 & 0 & -3.732 & -2.799 & 3.732 & -2.799 \\ 0 & 0 & 2.799 & 1.4 & -2.799 & 2.799 \end{bmatrix}\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{Bmatrix}=\begin{Bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \\ F_3 \\ M_3 \end{Bmatrix}=\begin{Bmatrix} F_1 \\ M_1 \\ -20\times10^3 \\ M_2 \\ -20\times10^3 \\ M_3 \end{Bmatrix}$$

Applying boundary conditions,

$$v_1=\theta_1=0,$$

Deleteing 1st, 2nd row and column from the above equation,

$$10^7\begin{bmatrix} 11.196 & -2.799 & -3.732 & 2.799 \\ -2.799 & 8.397 & -2.799 & 1.4 \\ -3.732 & -2.799 & 3.732 & -2.799 \\ 2.799 & 1.4 & -2.799 & 2.799 \end{bmatrix}\begin{Bmatrix} v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{Bmatrix}=\begin{Bmatrix} -20\times10^3 \\ 0 \\ -20\times10^3 \\ 0 \end{Bmatrix}$$

On solving above equation,

$$\begin{Bmatrix} v_2 \\ \theta_2 \\ v_3 \\ \theta_4 \end{Bmatrix} = \begin{Bmatrix} -0.00375 \text{ m} \\ -0.00428 \text{ rad} \\ -0.01232 \text{ m} \\ -0.00642 \text{ rad} \end{Bmatrix}$$

$\therefore$ Nodal displacement vectors,

$$\{\delta\} = \{0 \quad 0 \quad -0.00375 \quad -0.00428 \quad -0.01232 \quad -0.00642\}^T$$

Gauss Elimination Method

to eliminate or remove variables

from ?

System of Linear Equations

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

Remove x

Remove y

Then find z

Second Method

Form augmented matrix [A | B]

$$\left[\begin{array}{ccc|c} \checkmark & \checkmark & \checkmark & \checkmark \\ 0 & \checkmark & \checkmark & \checkmark \\ 0 & 0 & \checkmark & \checkmark \end{array}\right]$$

Solution of Augmented Matrix

$\Downarrow$

A is upper triangular matrix

Example 28.13. *Apply Gauss elimination method to solve the equations $x + 4y - z = -5$; $x + y - 6z = -12$; $3x - y - z = 4$.* *(Mumbai, 2009)*

$$x + 4y - z = -5$$
$$x + y - 6z = -12$$
$$3x - y - z = 4$$

: We have $\begin{bmatrix} 1 & 4 & -1 \\ 1 & 1 & -6 \\ 3 & -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ -12 \\ 4 \end{bmatrix}$

Operate $R_2 - R_1$ and $R_3 - 3R_1$, $\begin{bmatrix} 1 & 4 & -1 \\ 0 & -3 & -5 \\ 0 & -13 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ -7 \\ 19 \end{bmatrix}$

Operate $R_3 - \frac{13}{3}R_2$, $\begin{bmatrix} 1 & 4 & -1 \\ 0 & -3 & -5 \\ 0 & 0 & 71/3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ -7 \\ 148/3 \end{bmatrix}$

Thus, we have $z = 148/71 = 2.0845$,

$$3y = 7 - 5z = 7 - 10.4225 = -3.4225 \quad i.e., \quad y = -1.1408$$
$$x = -5 - 4y + z = -5 + 4\,(1.1408) + 2.0845 = 1.6479$$

Hence $x = 1.6479$, $y = -1.1408$, $z = 2.0845$.

Example 28.13. *Apply Gauss elimination method to solve the equations $x + 4y - z = -5$; $x + y - 6z = -12$; $3x - y - z = 4$.* (*Mumbai, 2009*)

$$x + 4y - z = -5$$
$$x + y - 6z = -12$$
$$3x - y - z = 4$$

: We have $\begin{bmatrix} 1 & 4 & -1 \\ 1 & 1 & -6 \\ 3 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ -12 \\ 4 \end{bmatrix}$

Operate $R_2 - R_1$ and $R_3 - 3R_1$, $\begin{bmatrix} 1 & 4 & -1 \\ 0 & -3 & -5 \\ 0 & -13 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ -7 \\ 19 \end{bmatrix}$

Operate $R_3 - \frac{13}{3} R_2$, $\begin{bmatrix} 1 & 4 & -1 \\ 0 & -3 & -5 \\ 0 & 0 & 71/3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ -7 \\ 148/3 \end{bmatrix}$

Thus, we have $z = 148/71 = 2.0845$,

$$3y = 7 - 5z = 7 - 10.4225 = -3.4225 \quad i.e., \quad y = -1.1408$$
$$x = -5 - 4y + z = -5 + 4\,(1.1408) + 2.0845 = 1.6479$$

Hence $x = 1.6479$, $y = -1.1408$, $z = 2.0845$.

Deleteing 1st, 2nd row and column from the above equation,

$$10^7\begin{bmatrix} 11.196 & -2.799 & -3.732 & 2.799 \\ -2.799 & 8.397 & -2.799 & 1.4 \\ -3.732 & -2.799 & 3.732 & -2.799 \\ 2.799 & 1.4 & -2.799 & 2.799 \end{bmatrix}\begin{Bmatrix} v_2 \\ \theta_2 \\ v_3 \\ \theta_4 \end{Bmatrix} = \begin{Bmatrix} -20\times10^3 \\ 0 \\ -20\times10^3 \\ 0 \end{Bmatrix}$$

Rounding off the values

$$\left(\begin{array}{cccc|c} 11 & -3 & -4 & 3 & -2 \\ -3 & 8 & -3 & 1 & 0 \\ -4 & -3 & 4 & -3 & -2 \\ 3 & 1 & -3 & 3 & 0 \end{array}\right)$$

R2=R2+R4

$$\left(\begin{array}{cccc|c} 11 & -3 & -4 & 3 & -2 \\ 0 & 9 & -6 & 4 & 0 \\ -4 & -3 & 4 & -3 & -2 \\ 3 & 1 & -3 & 3 & 0 \end{array}\right)$$

R3=11*R3 + 4*R1

$$\left(\begin{array}{cccc|c} 11 & -3 & -4 & 3 & -2 \\ 0 & 9 & -6 & 4 & 0 \\ 0 & -5 & 0 & 3 & -6 \\ 3 & 1 & -3 & 3 & 0 \end{array}\right)$$

R4=11*R4 - 3*R1

$$\left(\begin{array}{cccc|c} 11 & -3 & -4 & 3 & -2 \\ 0 & 9 & -6 & 4 & 0 \\ 0 & -5 & 0 & 3 & -6 \\ 0 & 20 & -21 & 24 & 6 \end{array}\right)$$

R3=9*R3 + 5*R2

$$\left(\begin{array}{cccc|c} 11 & -3 & -4 & 3 & -2 \\ 0 & 9 & -6 & 4 & 0 \\ 0 & 0 & -30 & 47 & -54 \\ 0 & 20 & -21 & 24 & 6 \end{array}\right)$$

R4=9*R4 - 20*R2

$$\left(\begin{array}{cccc|c} 11 & -3 & -4 & 3 & -2 \\ 0 & 9 & -6 & 4 & 0 \\ 0 & 0 & -30 & 47 & -54 \\ 0 & 0 & -69 & 136 & 54 \end{array}\right)$$

R3= -R3
R4= -R4

$$\left(\begin{array}{cccc|c} 11 & -3 & -4 & 3 & -2 \\ 0 & 9 & -6 & 4 & 0 \\ 0 & 0 & 30 & -47 & 54 \\ 0 & 0 & 69 & -136 & -54 \end{array}\right)$$

R4=30*R4 - 69*R3

$$\left(\begin{array}{cccc|c} 11 & -3 & -4 & 3 & -2 \\ 0 & 9 & -6 & 4 & 0 \\ 0 & 0 & 30 & -47 & 54 \\ 0 & 0 & 0 & -837 & -5346 \end{array}\right)$$

$$10^3\begin{bmatrix} 11 & -3 & -4 & 3 \\ 0 & 9 & -6 & 4 \\ 0 & 0 & 30 & -47 \\ 0 & 0 & 0 & -837 \end{bmatrix}\begin{Bmatrix} v_2 \\ \theta_2 \\ v_3 \\ \theta_4 \end{Bmatrix} = \begin{Bmatrix} -2 \\ 0 \\ 54 \\ -5346 \end{Bmatrix}$$

On writing each equation separately one can calculate the unknowns

$[K]\{\delta\} = \{F\}$

$$10^7 \begin{bmatrix} \boxed{\begin{matrix} 7.464 & 5.598 & -7.464 & 5.598 & 0 & 0 \\ 5.598 & 5.598 & -5.598 & 2.799 & 0 & 0 \end{matrix}} \\ \begin{matrix} -7.464 & -5.598 & 11.196 & -2.799 & -3.732 & 2.799 \\ 5.598 & 2.799 & -2.799 & 8.397 & -2.799 & 1.4 \\ 0 & 0 & -3.732 & -2.799 & 3.732 & -2.799 \\ 0 & 0 & 2.799 & 1.4 & -2.799 & 2.799 \end{matrix} \end{bmatrix} \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \\ F_3 \\ M_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ M_1 \\ -20\times10^3 \\ M_2 \\ -20\times10^3 \\ M_3 \end{Bmatrix}$$

M2, M3 are zero
From the diagram

$$F_1 = 10^7\,[-7.464\, v_2 + 5.598 \times \theta_2]$$
$$= 10^7\,[(7.464 \times 0.00375) - (5.598 \times 0.00428)]$$
$$F_1 = 40305.6 \text{ N}$$
$$M_1 = 10^7\,[-5.598 \times v_2 + 2.799 \times \theta_2]$$
$$= 10^7\,[(5.598 \times 0.00375) - (2.799 \times 0.00428)]$$
$$= 10^7 \times 0.00901 \text{ N-m} = 90127.8 \text{ N-m}$$

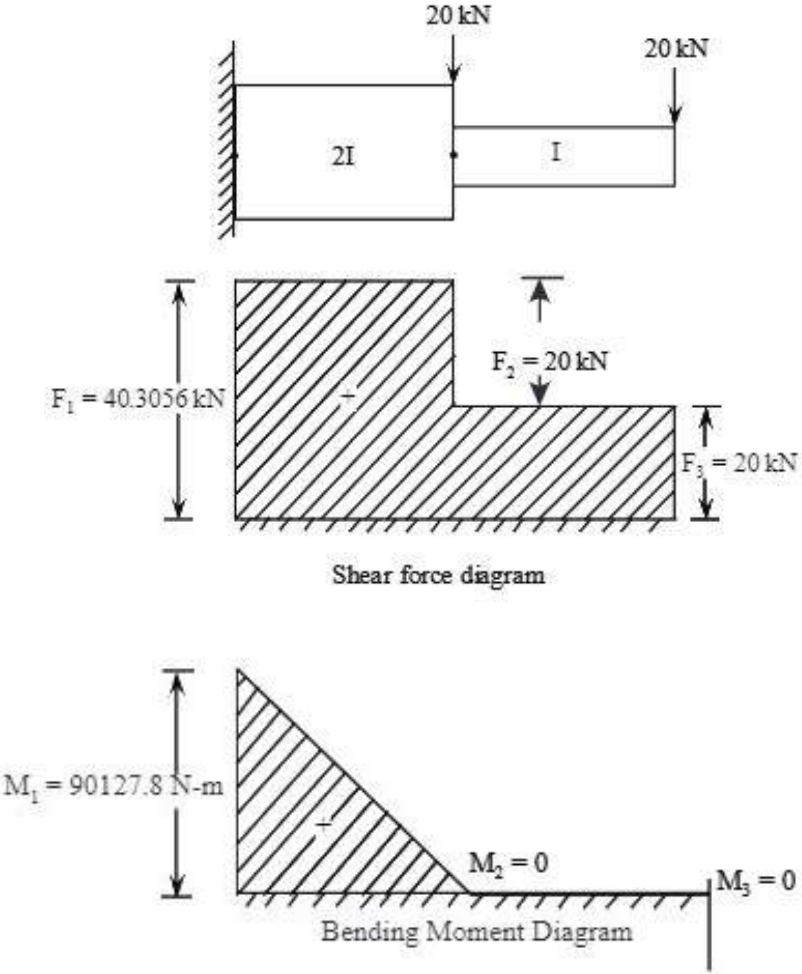
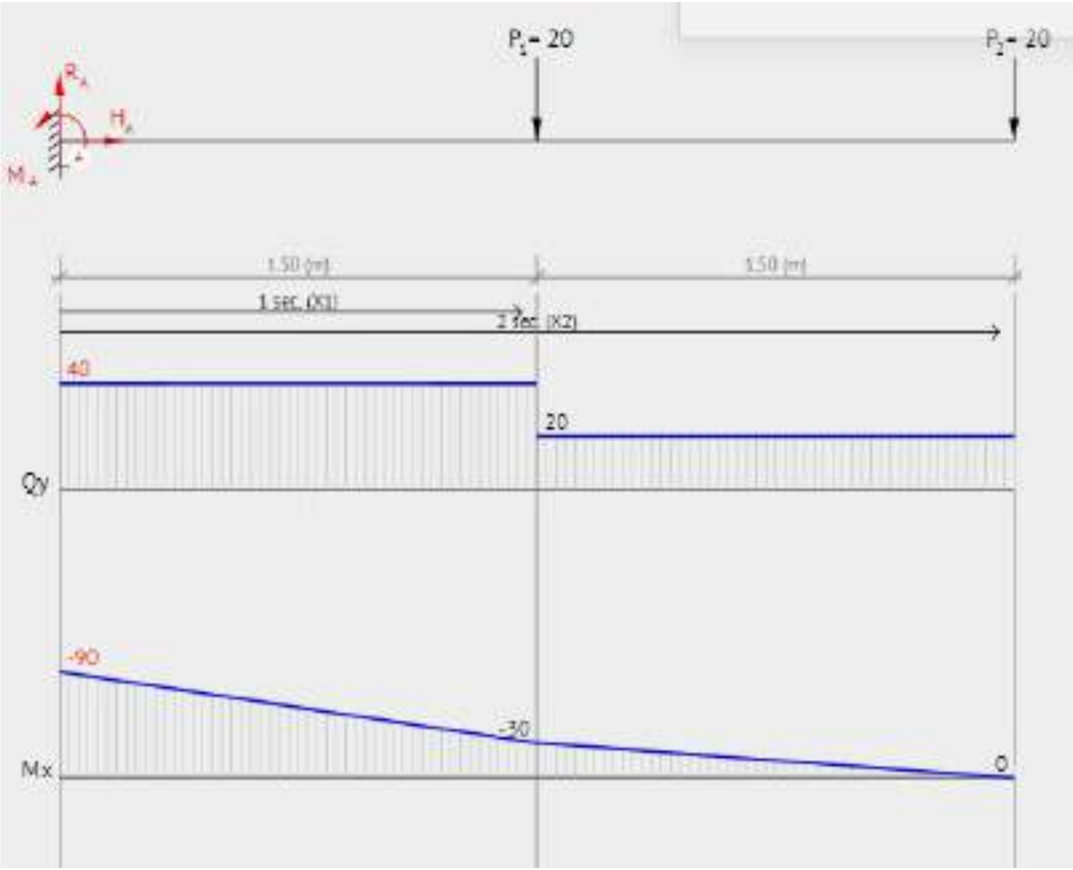
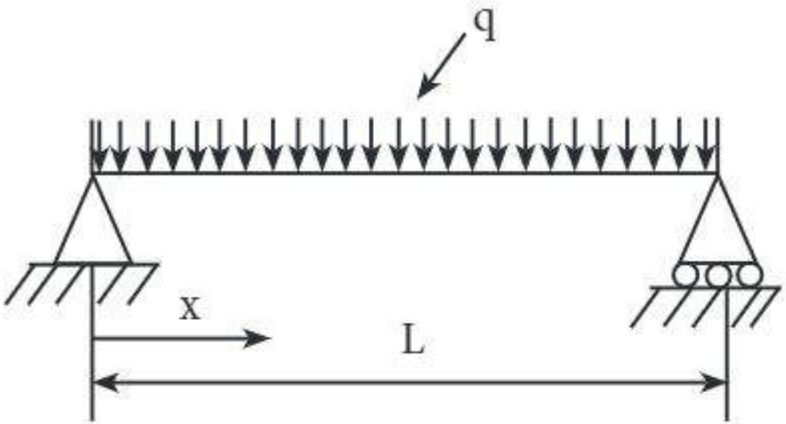


Figure (2): SF and BM diagrams



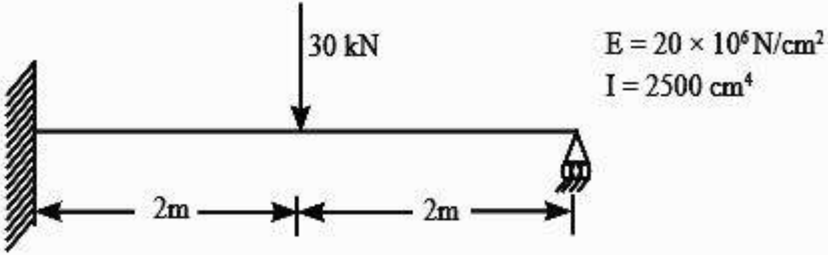
https://beamguru.com/online/beam-calculator/

Determine the maximum deflection and slope for the simple supported beam subjected to uniformly load ‘q’ as shown in Figure.



Figure

Q. For the beam shown in figure calculate the deflection under the load for the beam.



Figure

Given that,

Young's modulus of the beam material, $E = 20 \times 10^6$ N/cm^2

Moment of inertia, $I = 2500$ cm^4

Point load, W = 30 kN = 30000 N

Length of each element, $l_1 = l_2 = 2\text{m} = 200$ cm.

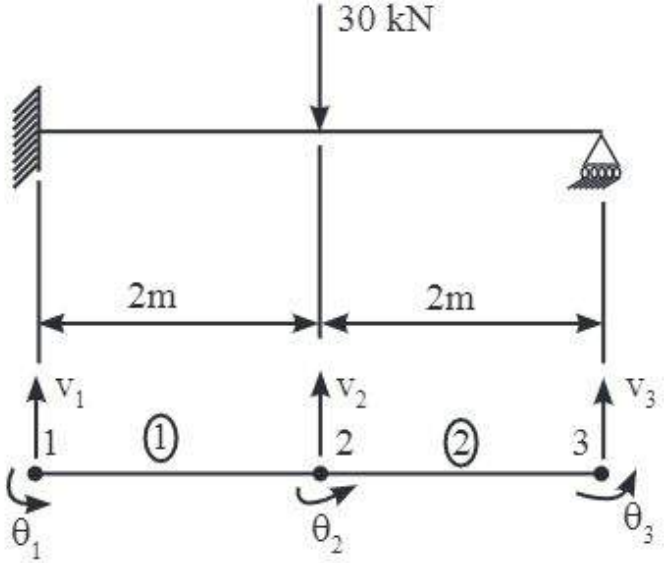


Figure: Discretization of Beam

Let v_1, θ_1, v_2, θ_2, v_3, θ_3 are the nodal displacements.

Element stiffness matrix,

$$[K] = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$$

For element – 1

Stiffness matrix for element (1) is given by,

$$[K_1] = \frac{E_1 I_1}{l_1^3}\begin{bmatrix} 12 & 6l_1 & -12 & 6l_1 \\ 6l_1 & 4l_1^2 & -6l_1 & 2l_1^2 \\ -12 & -6l_1 & 12 & -6l_1 \\ 6l_1 & 2l_1^2 & -6l_1 & 4l_1^2 \end{bmatrix}$$

$$[K_1] = \frac{20\times10^6\times2500}{(200)^3}\begin{bmatrix} 12 & 1200 & -12 & 1200 \\ 1200 & 160000 & -1200 & 80000 \\ -12 & -1200 & 12 & -1200 \\ 1200 & 80000 & -1200 & 160000 \end{bmatrix} = 6250 \begin{array}{c} \begin{array}{cccc} v_1 & \theta_1 & v_2 & \theta_2 \end{array} \\ \begin{bmatrix} 12 & 1200 & -12 & 1200 \\ 1200 & 160000 & -1200 & 80000 \\ -12 & -1200 & 12 & -1200 \\ 1200 & 80000 & -1200 & 160000 \end{bmatrix} \end{array} \begin{array}{c} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{array}$$

For element – 2

Stiffness matrix for element (2) is given by,

$$[K_2] = \frac{E_2 I_2}{l_2^3}\begin{bmatrix} 12 & 6l_2 & -12 & 6l_2 \\ 6l_2 & 4l_2^2 & -6l_2 & 2l_2^2 \\ -12 & -6l_2 & 12 & -6l_2 \\ 6l_2 & 2l_2^2 & 6l_2 & 4l_2^2 \end{bmatrix}$$

$$[K_2] = 6250 \begin{array}{c} \begin{array}{cccc} v_2 & \theta_2 & v_3 & \theta_3 \end{array} \\ \begin{bmatrix} 12 & 1200 & -12 & 1200 \\ 1200 & 160000 & -1200 & 80000 \\ -12 & -1200 & 12 & -1200 \\ 1200 & 80000 & -1200 & 160000 \end{bmatrix} \end{array} \begin{array}{c} \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{array}$$

$$[K_1] = 6250 \begin{array}{c} \begin{array}{cccc} v_1 & \theta_1 & v_2 & \theta_2 \end{array} \\ \begin{bmatrix} 12 & 1200 & -12 & 1200 \\ 1200 & 160000 & -1200 & 80000 \\ -12 & -1200 & 12 & -1200 \\ 1200 & 80000 & -1200 & 160000 \end{bmatrix} \end{array} \begin{array}{c} \\ v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{array}$$

$$[K_2] = 6250 \begin{array}{c} \begin{array}{cccc} v_2 & \theta_2 & v_3 & \theta_3 \end{array} \\ \begin{bmatrix} 12 & 1200 & -12 & 1200 \\ 1200 & 160000 & -1200 & 80000 \\ -12 & -1200 & 12 & -1200 \\ 1200 & 80000 & -1200 & 160000 \end{bmatrix} \end{array} \begin{array}{c} \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{array}$$

Global Stiffness Matrix

$$[K] = [K_1] + [K_2]$$

$$[K] = 6250 \times \begin{array}{c} \begin{array}{cccccc} v_1 & \theta_1 & v_2 & \theta_2 & v_3 & \theta_3 \end{array} \\ \begin{bmatrix} 12 & 1200 & -12 & 1200 & 0 & 0 \\ 1200 & 160000 & -1200 & 80000 & 0 & 0 \\ -12 & -1200 & 24 & 0 & -12 & 1200 \\ 1200 & 80000 & 0 & 320000 & -1200 & 80000 \\ 0 & 0 & -12 & -1200 & 12 & -1200 \\ 0 & 0 & 1200 & 80000 & -1200 & 160000 \end{bmatrix} \end{array} \begin{array}{c} \\ v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{array}$$

The finite element equation is given by,

$$[K]\{\delta\} = \{F\}$$

$$6250 \times \begin{bmatrix} 12 & 1200 & -12 & 1200 & 0 & 0 \\ 1200 & 160000 & -1200 & 80000 & 0 & 0 \\ -12 & -1200 & 24 & 0 & -12 & 1200 \\ 1200 & 80000 & 0 & 320000 & -1200 & 80000 \\ 0 & 0 & -12 & -1200 & 12 & -1200 \\ 0 & 0 & 1200 & 80000 & -1200 & 160000 \end{bmatrix} \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -30000 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Applying boundary conditions;

$$v_1 = \theta_1 = v_3 = 0$$

The rows and columns related to degree of freedom 1, 2 and 5 are deleted from $[K]$ matrix.

$$\therefore 6250 \times \begin{bmatrix} 24 & 0 & 1200 \\ 0 & 320000 & 80000 \\ 1200 & 80000 & 160000 \end{bmatrix} \begin{Bmatrix} v_2 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} -30000 \\ 0 \\ 0 \end{Bmatrix}$$

On solving above equations,

$$24v_2 + 1200\theta_3 = -4.8 \Rightarrow v_2 = \left(\frac{-4.8 - 1200\theta_3}{24}\right)$$

$$3200000\theta_2 + 800000\theta_3 = 0 \Rightarrow \theta_2 = -\frac{\theta_3}{4}$$

$$1200v_2 + 80000\theta_2 + 160000\theta_3 = 0$$

$$1200\left[\frac{-4.8 - 1200\theta_3}{24}\right] + 80000\left[\frac{-\theta_3}{4}\right] + 160000\theta_3 = 0$$

$$-240 - 60000\theta_3 - 20000\theta_3 + 160000\theta_3 = 0$$

$$\therefore \theta_3 = 0.003$$

$$v_2 = \frac{-4.8 - 1200(0.003)}{24} = -0.35 \text{ cm}$$

$$\theta_2 = \frac{-0.003}{4} = -0.00075 \text{ rad}$$

The nodal displacement vector is given by,

$$\{\delta\} = [0 \quad 0 \quad -0.35 \text{ cm} \quad -0.00075 \text{ rad} \quad 0 \quad 0.003 \text{ cm}]^T$$

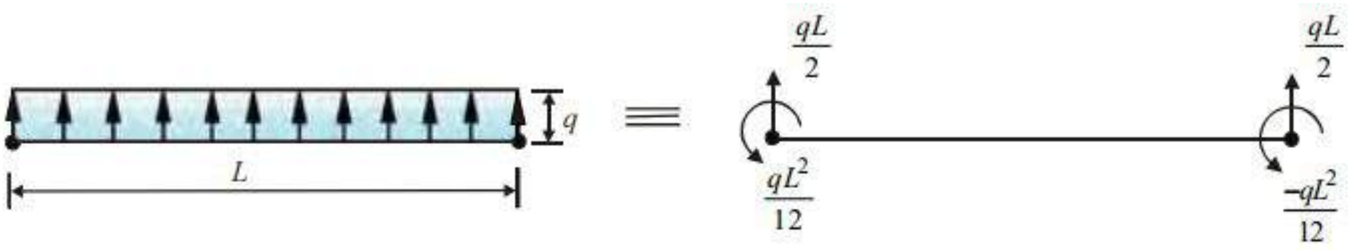
Gravity loading

Gravity loading is a typical body force and is given by (ρg) per unit volume or (ρAg) per unit length, where ρ is the mass density of the material. The equivalent nodal force vector for the distributed body force can be obtained as

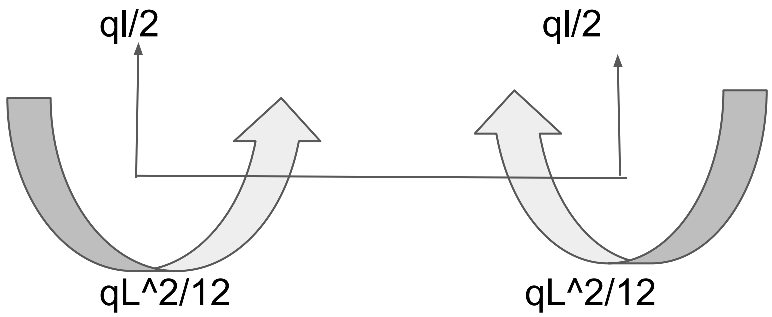
$$W_q = \int_0^L qv\, ds$$

$$\{f\}^e = \int_v [N]^T (\rho g)\, dv = \int_0^L [N]^T (\rho g) A\, dx = (\rho Ag) \begin{Bmatrix} \ell/2 \\ L^2/12 \\ \ell/2 \\ -L^2/12 \end{Bmatrix} \tag{4.124}$$

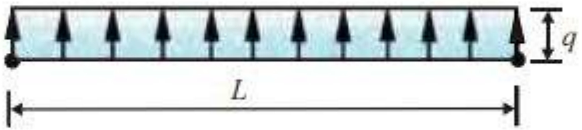
$$\boxed{\{f\}^e = \int_0^L [N]^T q_0\, dx = \begin{Bmatrix} q_0 L/2 \\ q_0 L^2/12 \\ q_0 L/2 \\ -q_0 L^2/12 \end{Bmatrix}}$$

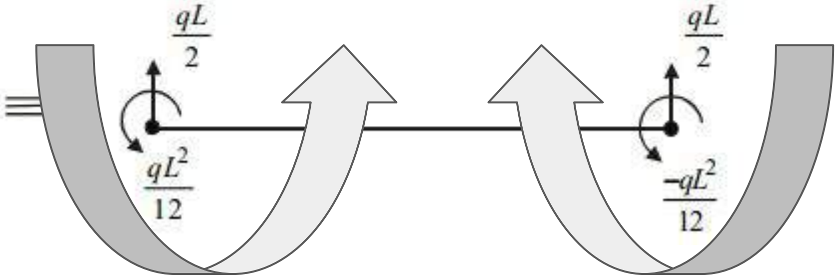


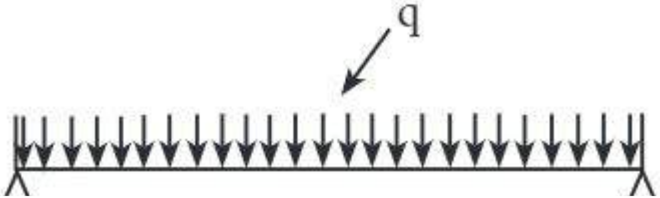
q
L
$\frac{qL}{2}$
$\frac{qL}{2}$
$\frac{qL^2}{12}$
$\frac{-qL^2}{12}$

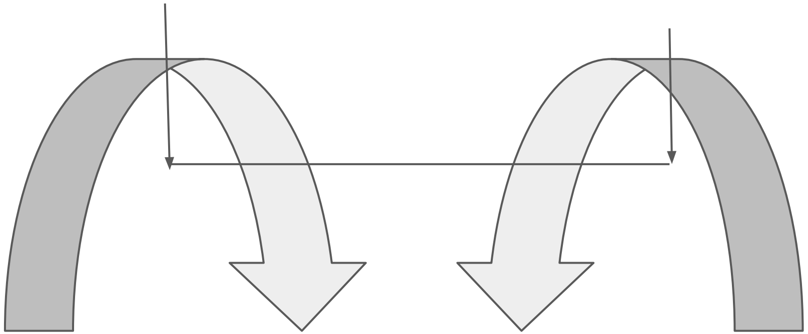


ql/2
ql/2
qL^2/12
qL^2/12


q
L


qL/2
qL/2
qL²/12
−qL²/12

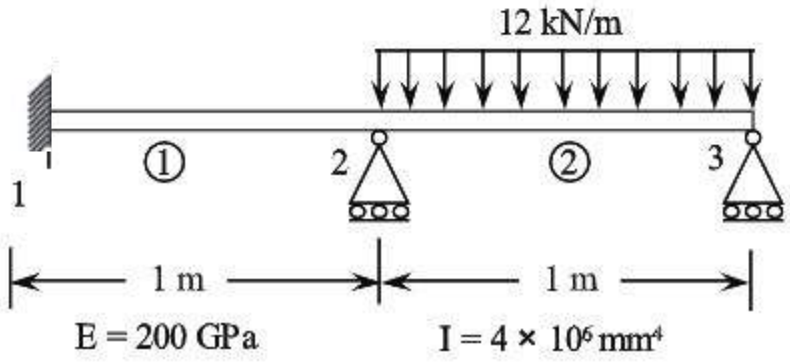

q

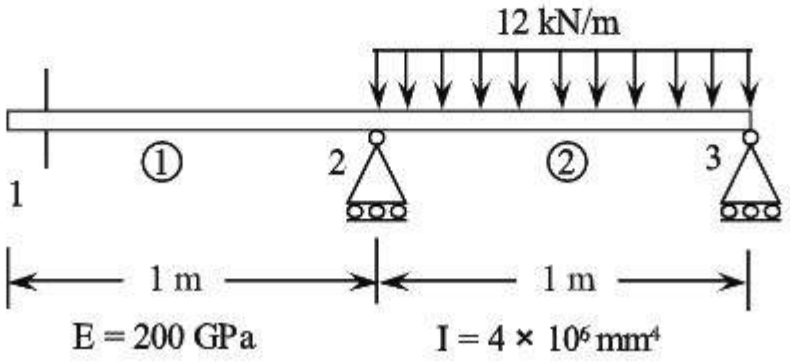


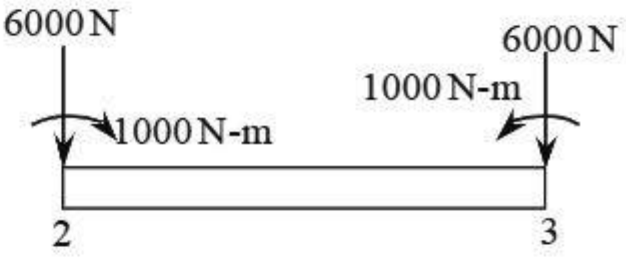
Q. For the beam and loading shown in the figure determine,

(i) The slopes at 2 and 3

(ii) The vertical deflection at the midpoint of the distributed load.




12 kN/m
1
①
2
②
3
1 m
1 m
E = 200 GPa
I = 4 × 10⁶ mm⁴


6000 N
6000 N
1000 N-m
1000 N-m
2
3

Figure

Given that,

Young's modulus, $E = 200 \text{ GPa} = 200 \times 10^9 \text{ N/m}^2$

Moment of inertia, $I = 4 \times 10^6 \text{ mm}^4 = 4 \times 10^{-6} \text{ m}^4$

Length of elements, $l_1 = l_2 = 1 \text{ m}$

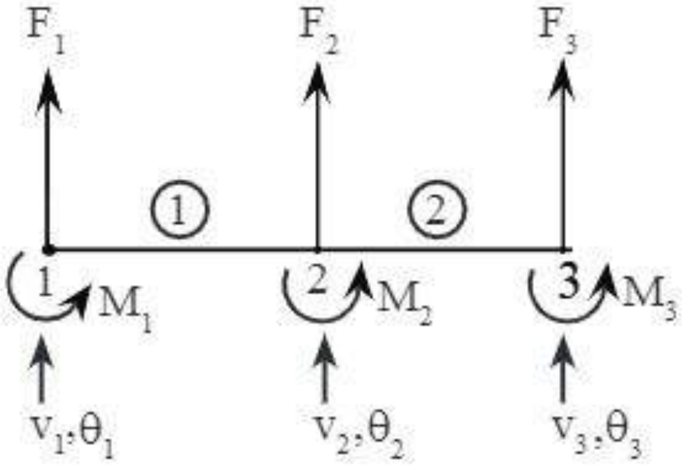


Figure (1): Discretization of Beam

Let, v_1, θ_1, v_2, θ_2, v_3, θ_3 are the nodal displacements.

Nodal displacement vector $\{\delta\}$ is given as,

$$\{\delta\} = \left[v_1 \;\; \theta_1 \;\; v_2 \;\; \theta_2 \;\; v_3 \;\; \theta_3\right]$$

Element stiffness matrix is given by,

$$[K] = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$$

For Element (1)

Stiffness matrix,

$$[K_1] = \frac{200\times10^9\times4\times10^{-6}}{1^3}\begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix}$$

$$[K_1] = 8\times10^5 \begin{array}{c} \begin{array}{cccc} v_1 & \theta_1 & v_2 & \theta_2 \end{array} \\ \begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix} \end{array} \begin{array}{c} \\ v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{array}$$

Element stiffness matrix is given by,

$$[K] = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$$

For Element (2)

Stiffness matrix,

$$[K_2] = \frac{200\times10^9\times4\times10^{-6}}{1^2}\begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix}$$

$$[K_2] = 8\times10^5 \begin{array}{c} \begin{array}{cccc} v_2 & \theta_2 & v_3 & \theta_3 \end{array} \\ \begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix} \end{array} \begin{array}{c} \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{array}$$

Then, the global stiffness matrix 'K' is given by,

$$[K] = [K_1] + [K_2]$$

$$= 8 \times 10^5 \begin{bmatrix} 12 & 6 & -12 & 6 & 0 & 0 \\ 6 & 4 & -6 & 2 & 0 & 0 \\ -12 & -6 & (12+12) & (-6+6) & -12 & 6 \\ 6 & 2 & (-6+6) & (4+4) & -6 & 2 \\ 0 & 0 & -12 & -6 & 12 & -6 \\ 0 & 0 & 6 & 2 & -6 & 4 \end{bmatrix}$$

$$[K] = 8 \times 10^5 \begin{array}{c} \begin{array}{cccccc} v_1 & \theta_1 & v_2 & \theta_2 & v_3 & \theta_3 \end{array} \\ \begin{bmatrix} 12 & 6 & -12 & 6 & 0 & 0 \\ 6 & 4 & -6 & 2 & 0 & 0 \\ -12 & -6 & 24 & 0 & -12 & 6 \\ 6 & 2 & 0 & 8 & -6 & 2 \\ 0 & 0 & -12 & -6 & 12 & -6 \\ 0 & 0 & 6 & 2 & -6 & 4 \end{bmatrix} \end{array} \begin{array}{c} \\ v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{array}$$

Finite element equation is given by,

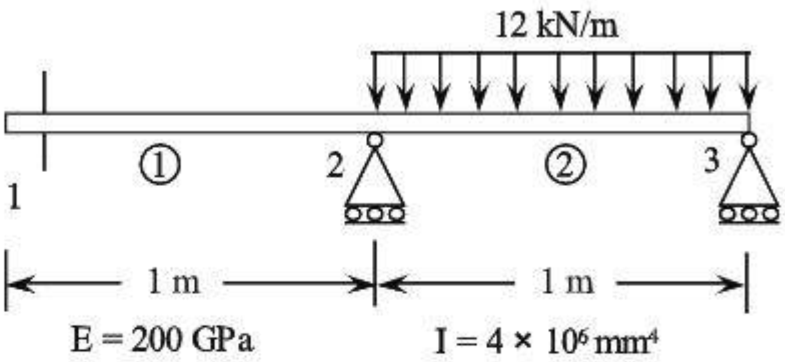
$$[K]\{\delta\} = \{F\}$$

$$8\times10^5\begin{bmatrix} 12 & 6 & -12 & 6 & 0 & 0 \\ 6 & 4 & -6 & 2 & 0 & 0 \\ -12 & -6 & 24 & 0 & -12 & 6 \\ 6 & 2 & 0 & 8 & -6 & 2 \\ 0 & 0 & -12 & -6 & 12 & -6 \\ 0 & 0 & 6 & 2 & -6 & 4 \end{bmatrix}\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} F_1 + F_{1d} \\ M_1 + M_{1d} \\ F_2 + F_{2d} \\ M_2 + M_{2d} \\ F_3 + F_{3d} \\ M_3 + M_{3d} \end{Bmatrix}$$

Following are due to UDL

$F_{1d}, F_{2d}, F_{3d} \rightarrow$ Nodal forces

$M_{1d}, M_{2d}, M_{3d} \rightarrow$ Nodal bending moments

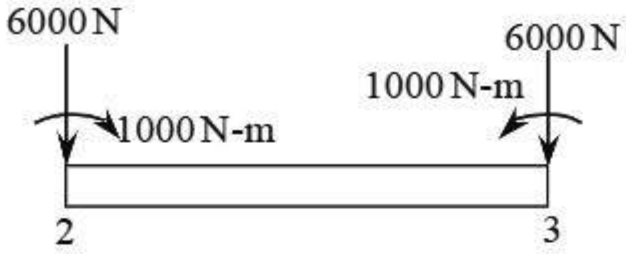


Applying the boundary conditions,

$$v_1 = 0,\ \theta_1 = 0,\ v_2 = 0,\ v_3 = 0,\ F_{1d} = 0,\ M_{1d} = 0,$$

$$F_{2d} = -6000\ \text{N},\ M_{2d} = -1000\ \text{N-m},\ F_{3d} = -6000\ \text{N},$$

$$M_{3d} = +1000\ \text{N-m},\ M_2 = 0,\ M_3 = 0$$



Figure

$$F_{2d} = F_{3d} = \frac{Wl}{2} = \frac{12000 \times 1}{2} = 6000\ \text{N}$$

$$M_{2d} = M_{3d} = \frac{Wl^2}{12} = \frac{12000 \times 1^2}{12} = 1000\ \text{N-m}$$

$$8\times10^{5}\begin{bmatrix}12 & 6 & -12 & 6 & 0 & 0\\ 6 & 4 & -6 & 2 & 0 & 0\\ -12 & -6 & 24 & 0 & -12 & 6\\ 6 & 2 & 0 & 8 & -6 & 2\\ 0 & 0 & -12 & -6 & 12 & -6\\ 0 & 0 & 6 & 2 & -6 & 4\end{bmatrix}\begin{Bmatrix}v_1\\ \theta_1\\ v_2\\ \theta_2\\ v_3\\ \theta_3\end{Bmatrix}=\begin{Bmatrix}F_1\\ M_1\\ F_2-6000\\ -1000\\ F_3-6000\\ 1000\end{Bmatrix}$$

$$8\times10^{5}\begin{bmatrix}8 & 2\\ 2 & 4\end{bmatrix}\begin{Bmatrix}\theta_2\\ \theta_3\end{Bmatrix}=\begin{Bmatrix}-1000\\ 1000\end{Bmatrix}$$

$$8\times10^{5}[8\theta_2+2\theta_3]=-1000$$

$$8\times10^{5}[2\theta_3+4\theta_3]=1000$$

On solving the above equations,

$$\theta_2=-2.679\times10^{-4}\text{ rad}$$

$$\theta_3=4.464\times10^{-4}\text{ rad}$$

$\therefore$ Nodal displacement vector,

$$[\delta]=\begin{bmatrix}0 & 0 & 0 & -2.679\times10^{-4} & 0 & 4.464\times10^{-4}\end{bmatrix}^{T}$$

Vertical Deflection at the Midpoint of the Distributed Load

Consider element (2)

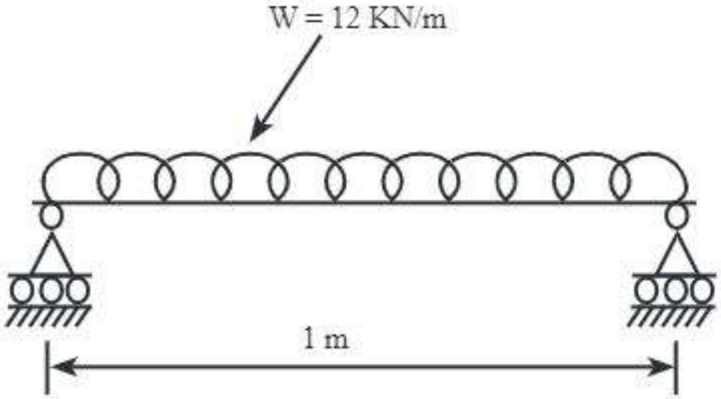


Figure (3): Element (2)

For a beam, element, the vertical deflection is given by,

$$V = N_1 v_1 + N_2 \theta_1 + N_3 v_2 + N_4 \theta_2 \qquad \text{... (1)}$$

Where,

N_1, N_2, N_3, N_4 – Shape functions

Consider element (2) as separate element and mark as node 1 and 2 as shown below,

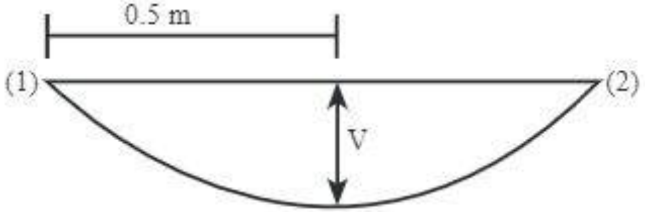


Figure (4)

Now,

$$v_1 = 0$$

$$\theta_1 = -2.679 \times 10^{-4} \text{ rad}$$

$$v_2 = 0$$

$$\theta_2 = 4.464 \times 10^{-4} \text{ rad}$$

$$N_1 v_1 = 0$$

$$N_2\theta_1 = \left[x - \frac{2x^2}{l} + \frac{x^3}{l^2}\right]\theta_1$$

$$= \left[0.5 - \frac{2(0.5)^2}{1} + \frac{(0.5)^3}{1^3}\right]\left(-2.679 \times 10^{-4}\right)$$

$$N_2\theta_1 = -3.348 \times 10^{-5}$$

$$N_2 v_2 = 0$$

$$N_4\theta_2 = \left(\frac{-x^2}{l} + \frac{x^3}{l^2}\right)\theta_2$$

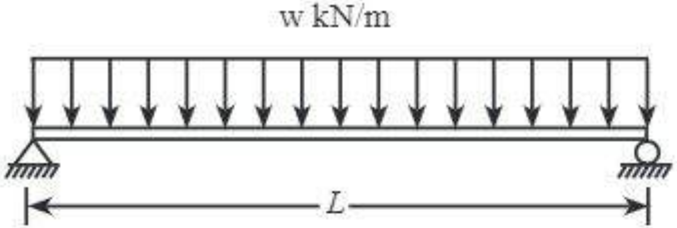
$$= \left(\frac{-(0.5)^2}{1} + \frac{(0.5)^3}{1^2}\right)\left(4.464 \times 10^{-4}\right)$$

$$N_4\theta_2 = -5.58 \times 10^{-5}$$

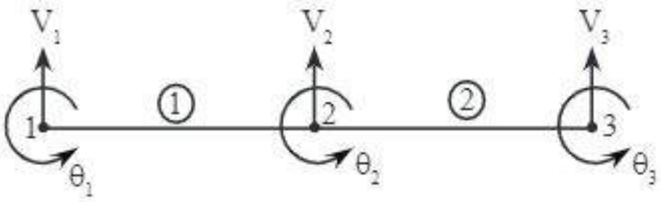
$$V = N_1 v_1 + N_2\theta_1 + N_3 v_2 + N_4\theta_2$$

$$V = 0 - 3.348 \times 10^{-5} + 0 - 5.58 \times 10^{-5}$$

$$V = -8.928 \times 10^{-5} \text{ m}$$

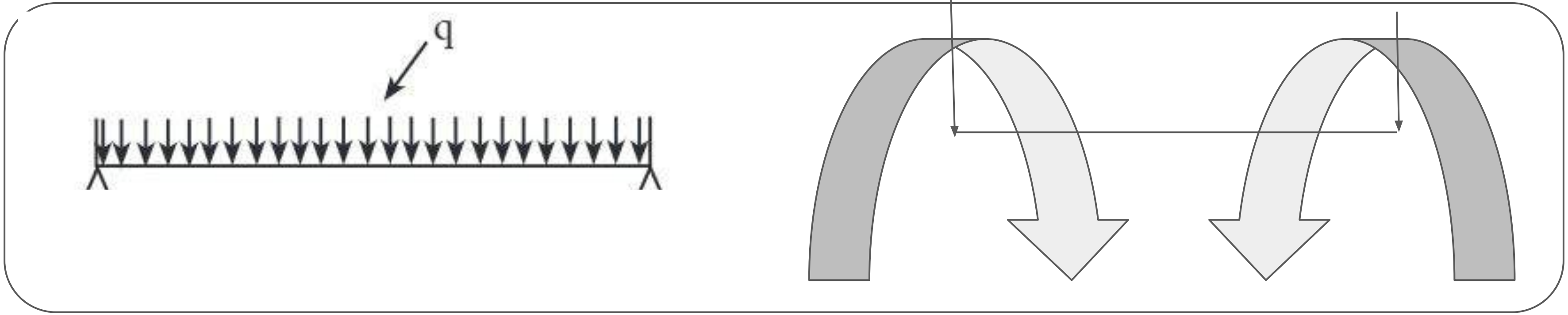


(i) Load Diagram



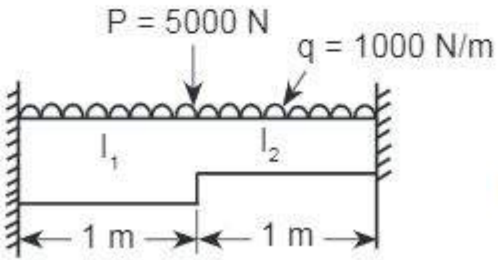
Force vector,

$$\{F\} = \begin{Bmatrix} -\dfrac{wL}{4} + R_1 \\ -\dfrac{wL^2}{48} \\ -\dfrac{wL}{2} \\ 0 \\ -\dfrac{wL}{4} + R_3 \\ \dfrac{wL^2}{48} \end{Bmatrix}$$

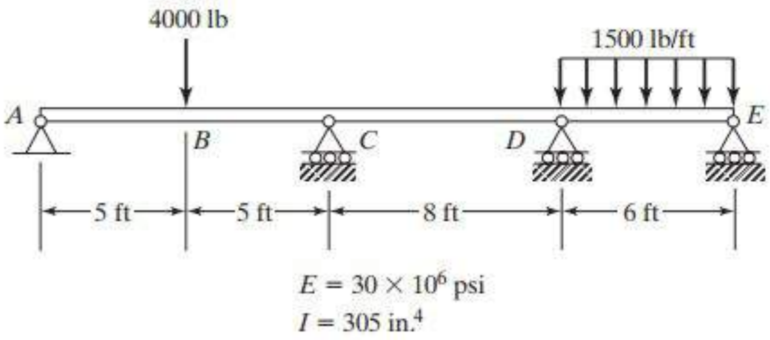


Determine the deflection in the beam, loaded as shown in figure, at the mid-span and at a length of 0.5 m from left support. Determine also the reactions at the fixed ends.

$E = 200$ GPa. $I_1 = 20 \times 10^{-6}$ m^4. $I_2 = 10 \times 10^{-6}$ m^4.

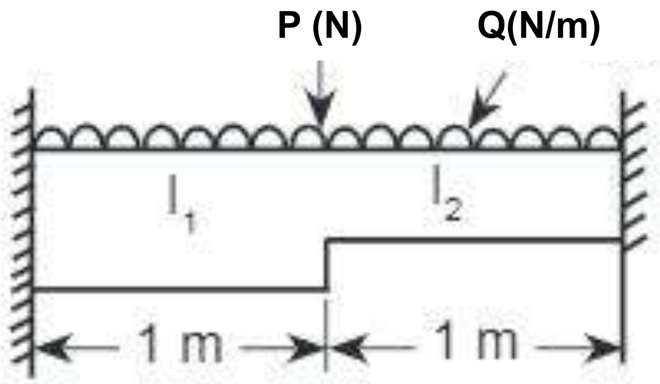


5.2. A three-span beam is shown in Fig. P5.2. Determine the deflection curve of the beam and evaluate the reactions at the supports.



2. Determine the deflection in the beam, loaded as shown in figure, at the mid-span and at a length of 0.5 m from left support. Determine also the reactions at the fixed ends.

$E = 200$ GPa. $I_1 = 20 \times 10^{-6}$ m^4. $I_2 = 10 \times 10^{-6}$ m^4.



Unit 3
Plane Problems (Two Dimensional Problems)

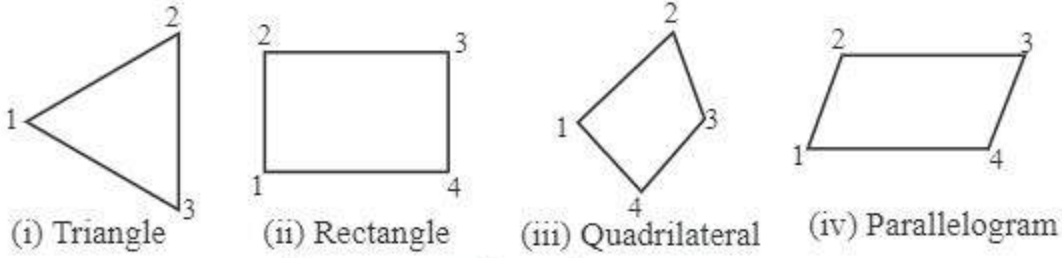


Figure: Two Dimensional Elements

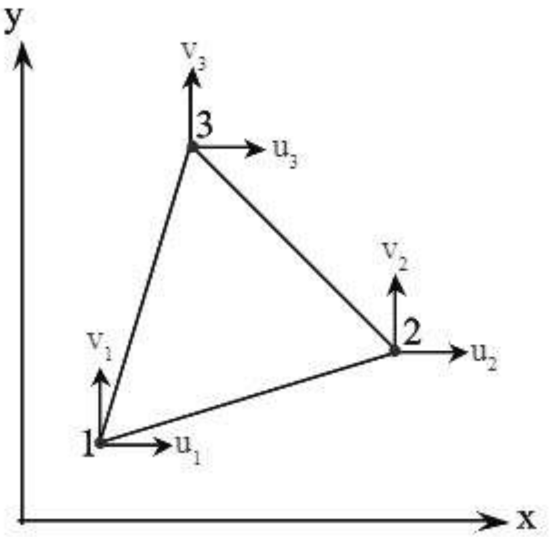


Figure (1): Constant Strain Triangle (CST)

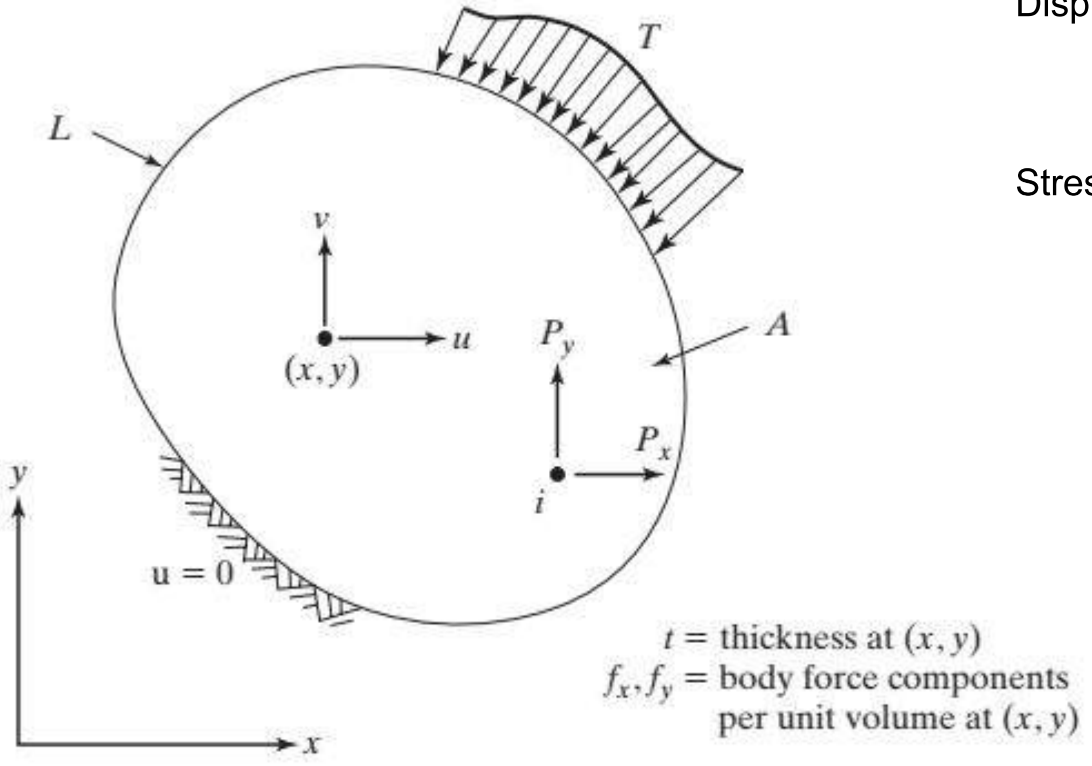


FIGURE 6.1 Two-dimensional problem.

Displacement vector

$$\mathbf{u} = [u \quad v]^{\mathrm{T}}$$

Stress and strains are,

$$\boldsymbol{\sigma} = [\sigma_x \quad \sigma_y \quad \tau_{xy}]^{\mathrm{T}}$$

$$\boldsymbol{\epsilon} = [\epsilon_x \quad \epsilon_y \quad \gamma_{xy}]^{\mathrm{T}}$$

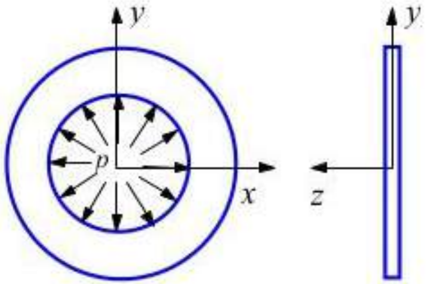
$$\mathbf{f} = [f_x \quad f_y]^{\mathrm{T}} \qquad \mathbf{T} = [T_x \quad T_y]^{\mathrm{T}} \qquad \text{and} \qquad dV = t\,dA$$

Plane (2-D) Problems

- *Plane stress*:

$$\sigma_z = \tau_{yz} = \tau_{zx} = 0 \qquad (\varepsilon_z \neq 0) \tag{1}$$

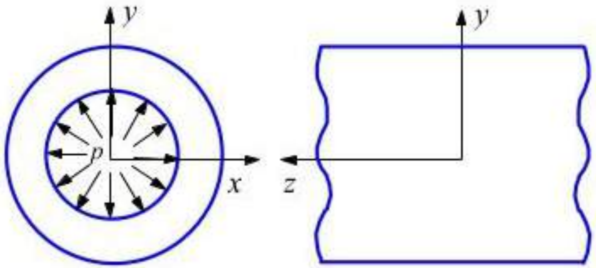
A thin planar structure with constant thickness and loading within the plane of the structure (*xy*-plane).



- *Plane strain*:

$$\varepsilon_z = \gamma_{yz} = \gamma_{zx} = 0 \qquad (\sigma_z \neq 0) \tag{2}$$

A long structure with a uniform cross section and transverse loading along its length (*z*-direction).



Stress-Strain-Temperature (Constitutive) Relations

For elastic and isotropic materials, we have,

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} 1/E & -\nu/E & 0 \\ -\nu/E & 1/E & 0 \\ 0 & 0 & 1/G \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} + \begin{Bmatrix} \varepsilon_{x0} \\ \varepsilon_{y0} \\ \gamma_{xy0} \end{Bmatrix} \quad (3)$$

where ε_0 is the initial strain, E the Young's modulus, ν the Poisson's ratio and G the shear modulus. Note that,

$$G = \frac{E}{2(1+\nu)} \quad (4)$$

which means that there are only two independent materials constants for *homogeneous* and *isotropic* materials.

We can also express stresses in terms of strains by solving the above equation,

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \left(\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} - \begin{Bmatrix} \varepsilon_{x0} \\ \varepsilon_{y0} \\ \gamma_{xy0} \end{Bmatrix} \right) \quad (5)$$

The above relations are valid for *plane stress* case. For *plane strain* case, we need to replace the material constants in the above equations in the following fashion,

$$
\begin{aligned}
&E \rightarrow \frac{E}{1-\nu^2} \\
&\nu \rightarrow \frac{\nu}{1-\nu} \\
&G \rightarrow G
\end{aligned}
\tag{6}
$$

For example, the stress is related to strain by

$$
\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & (1-2\nu)/2 \end{bmatrix} \left(\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} - \begin{Bmatrix} \varepsilon_{x0} \\ \varepsilon_{y0} \\ \gamma_{xy0} \end{Bmatrix} \right)
$$

in the *plane strain* case.

Initial strains due to *temperature change* (thermal loading) is given by,

$$
\begin{Bmatrix} \varepsilon_{x0} \\ \varepsilon_{y0} \\ \gamma_{xy0} \end{Bmatrix} = \begin{Bmatrix} \alpha\Delta T \\ \alpha\Delta T \\ 0 \end{Bmatrix}
\tag{7}
$$

where α is the coefficient of thermal expansion, ΔT the change of temperature. Note that if the structure is free to deform under thermal loading, there will be no (elastic) stresses in the structure.

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\epsilon}$$

$$\boldsymbol{\epsilon} = \left[\frac{\partial u}{\partial x} \;\; \frac{\partial v}{\partial y} \;\; \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right]^{\mathrm{T}}$$

Plane stress: $$\mathbf{D} = \frac{E}{1-\nu^2}\begin{pmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{pmatrix}$$

Plane strain: $$\mathbf{D} = \frac{E}{(1+\nu)(1-2\nu)}\begin{pmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{pmatrix}$$

Strain and Displacement Relations

For small strains and small rotations, we have,

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

In matrix form,

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \partial/\partial x & 0 \\ 0 & \partial/\partial y \\ \partial/\partial y & \partial/\partial x \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix}, \tag{8}$$

From this relation, we know that the strains (and thus stresses) are one order lower than the displacements, if the displacements are represented by polynomials.

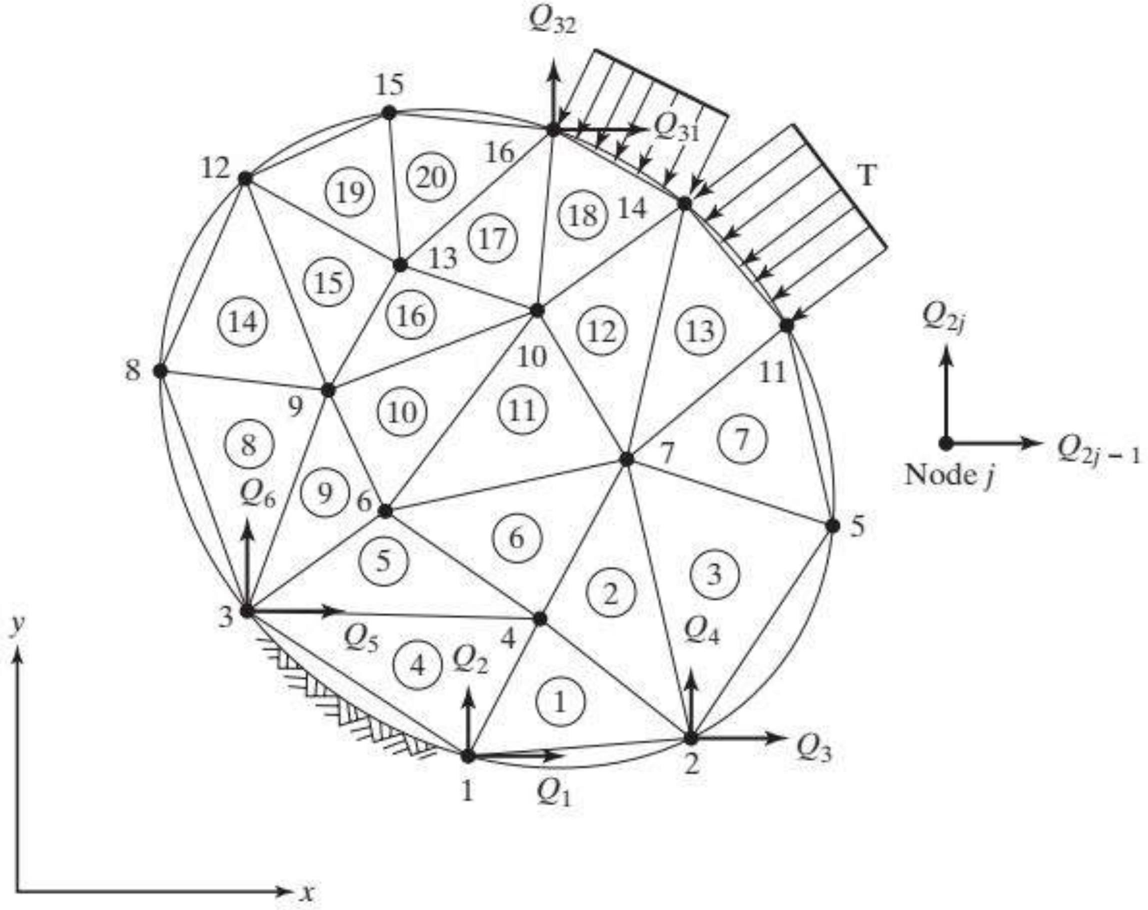


FIGURE 6.2 Finite element discretization.

Constant term (1) –			1		
Linear terms (2) –		x		y	
Quadratic terms (6) –		x^2	xy	y^2	
Cubic terms (10) –	x^3	x^2y		xy^2	y^3

Pascal triangle

In order to develop a polynomial with three terms,

Expression to be selected is,

$$u = a_1 + a_2x + a_3y$$

In order to develop a polynomial with four terms,

Expression to be selected is,

$$u = a_1 + a_2x + a_3y + a_4xy$$

In order to develop a polynomial with six terms,

Expression to be selected is,

$$u = a_1 + a_2x + a_3y + a_4x^2 + a_5xy + a_6y^2$$

LINEAR Displacement TRIANGLE or (CST)

A linear triangle is a plane triangle whose field quantity varies linearly with Cartesian coordinates x and y. In stress analysis, a linear displacement field produces a constant strain field, so the element may be called a constant-strain triangle (CST).

1. **Constant Strain Triangle (CST):** It consists of three nodes and six unknown nodal displacements. Its field varies linearly with coordinates x and y, giving rise to a linear displacement and a constant strain field.

Let, at a particular node,

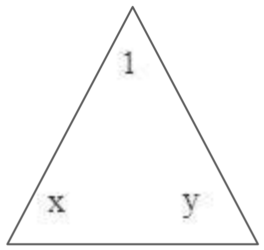
u – Displacement along x-axis

v – Displacement along y-axis.

Then, components of displacement for CST element are given by,

$$u = a_1 + a_2x + a_3y$$

$$v = a_4 + a_5x + a_6y$$



x^2 xy y^2

x^3 x^2y xy^2 y^3

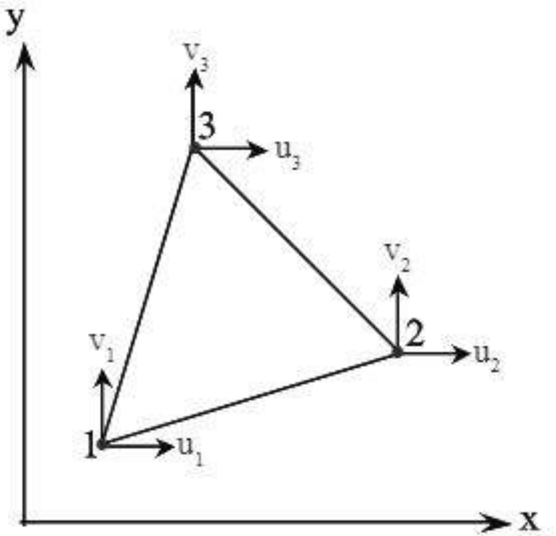


Figure (1): Constant Strain Triangle (CST)

$$u = a_1 + a_2 x + a_3 y$$
$$v = a_4 + a_5 x + a_6 y$$

$$u = \lfloor 1 \quad x \quad y \rfloor \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} \qquad v = \lfloor 1 \quad x \quad y \rfloor \begin{Bmatrix} a_4 \\ a_5 \\ a_6 \end{Bmatrix}$$

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \varepsilon_y = \frac{\partial v}{\partial y} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\varepsilon_x = a_2 \qquad \varepsilon_y = a_6 \qquad \gamma_{xy} = a_3 + a_5$$

$$\boxed{\begin{gathered} \varepsilon_x = \frac{\partial u}{\partial x} \qquad \varepsilon_y = \frac{\partial v}{\partial y} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\ \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} \end{gathered}}$$

QUADRATIC Displacement TRIANGLE or (LST)_____ displacement field varies quadratically

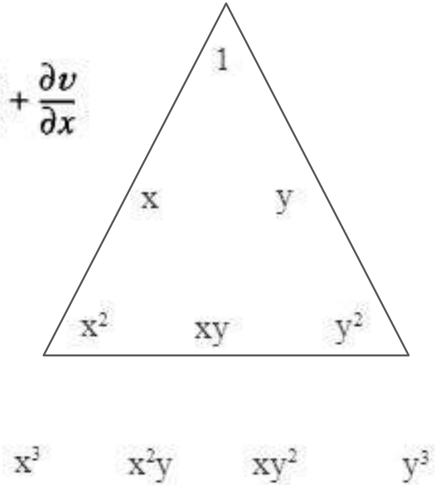
2. **Linear Strain Triangle (LST):** It consists of three primary nodes and three secondary nodes at the mid-points of the sides of the triangle. Each node possess two degrees of freedom (DOF). Therefore each element has 12 DOF. Displacement function for this element is a quadratic equation and strain field varies linearly.

Components of displacement for LST element are given by,

$$u = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 xy + a_6 y^2$$

$$v = a_7 + a_8 x + a_9 y + a_{10} x^2 + a_{11} xy + a_{12} y^2$$

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \varepsilon_y = \frac{\partial v}{\partial y} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$



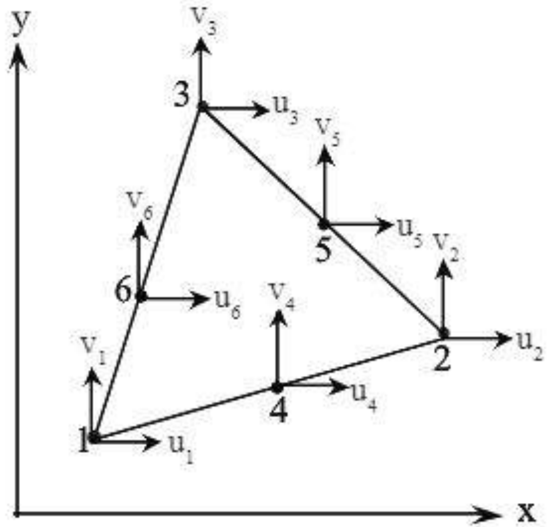


Figure (2): Linear Strain Triangle (LST)

<u>CST element:</u> (constant strains, linear displacements)

shape functions N_1, N_2 and N_3. Variation of these shape functions occurs linearly in CST element. These shape functions have a value of unity at their corresponding nodes and reduce to the value of zero at other nodes. Only two of the shape functions are independent and are represented by ξ and η in natural coordinate system.

$$N_1 = \xi,\ N_2 = \eta,\ N_3 = 1 - \xi - \eta$$

$$\therefore \quad N_1 + N_2 + N_3 = 1$$

In two dimensional problem, the x-y coordinates are represented by ξ and η coordinates in natural coordinate system.

α alpha	β beta	γ gamma	δ delta
ε epsilon	ζ zeta	η eta	θ theta
ι iota	κ kappa	λ lambda	μ mu
ν nu	ξ xi	o omicron	π pi
ρ rho	σ sigma	τ tau	υ upsilon
ϕ phi	χ chi	ψ psi	ω omega

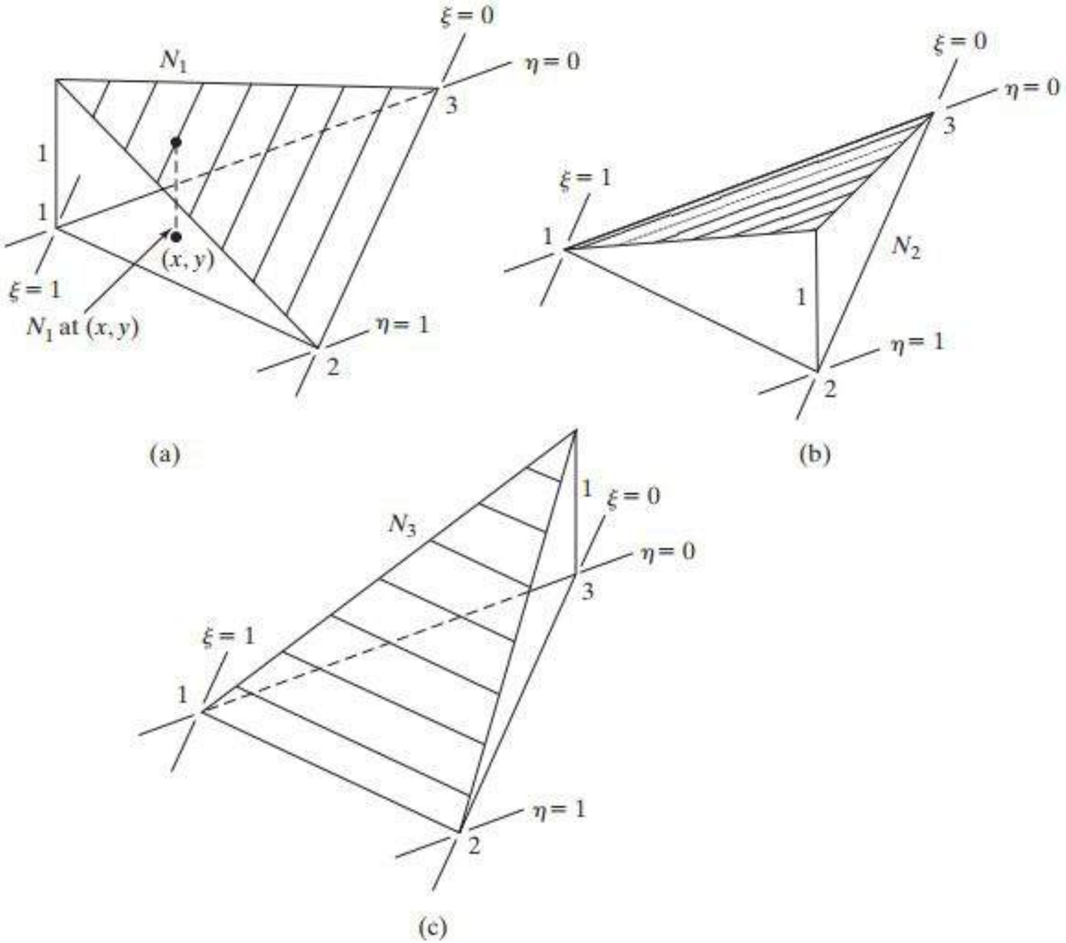


FIGURE 6.4 Shape functions.

Fig. (a) i.e, at node-1,

$$N_1 = 1$$

$$N_2 = N_3 = 0$$

Fig. (b) At node-2,

$$N_2 = 1$$

$$N_1 = N_3 = 0$$

Fig. (c) At node-3,

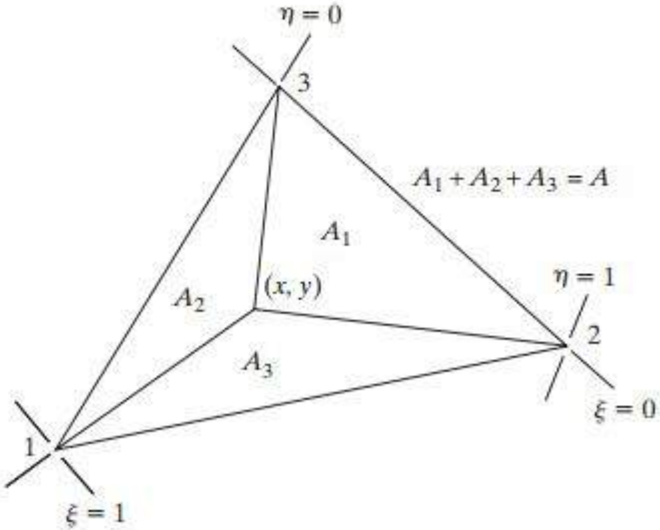
$$N_3 = 1$$

$$N_2 = N_1 = 0$$

The shape functions can be physically represented by **area coordinates**. A point (x, y) in a triangle divides it into three areas, A_1, A_2, and A_3, as shown in Fig. 6.5. The shape functions N_1, N_2, and N_3 are precisely represented by

$$N_1 = \frac{A_1}{A} \qquad N_2 = \frac{A_2}{A} \qquad N_3 = \frac{A_3}{A} \tag{6.11}$$

where A is the area of the element. Clearly, $N_1 + N_2 + N_3 = 1$ at all points inside the triangle.



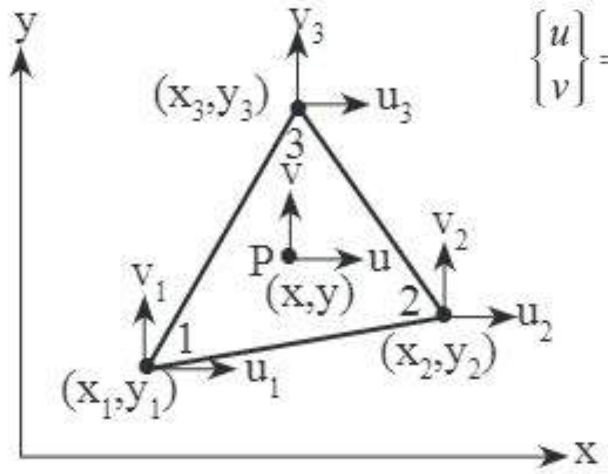
The displacements inside the element are now written using the shape functions and the nodal values of the unknown displacement field.

The displacements can be written as,

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3$$

$$v = N_1 v_1 + N_2 v_2 + N_3 v_3$$

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$



Figure

This is in the form of,

$$u = N\delta$$

Where,

$$N = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix}$$

$$u = \begin{Bmatrix} u \\ v \end{Bmatrix}$$ and

$$\delta = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

Isoparametric Representation

It is possible to represent the coordinates (x, y) of any point 'P' within the linear triangular element in terms of nodal coordinates by employing the same shape functions used to represent displacements u and v. Such a method of representation is known as 'isoparametric representation '.

Thus, coordinates (x, y) of point 'P',

$$N_1 = \xi$$

$$N_2 = \eta$$

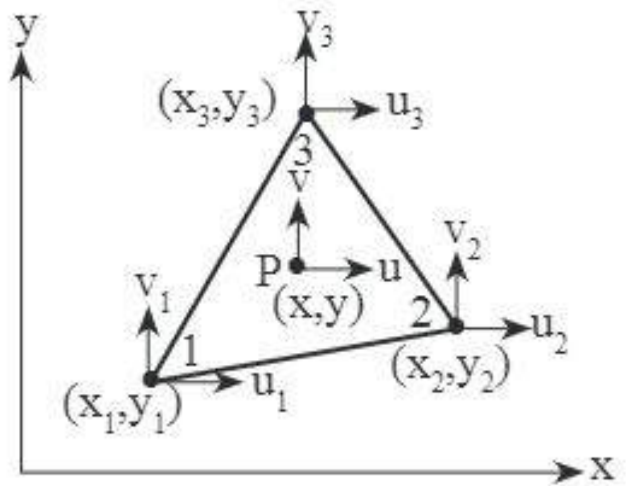
$$N_3 = 1 - \xi - \eta$$

i.e.,
$$\begin{aligned} x &= N_1 x_1 + N_2 x_2 + N_3 x_3 \\ &= \xi x_1 + \eta x_2 + (1 - \xi - \eta) x_3 \\ &= (x_1 - x_3)\xi + (x_2 - x_3)\eta + x_3 \end{aligned}$$

$$\therefore \quad x = x_{13}\xi + x_{23}\eta + x_3$$

$$\begin{aligned} y &= N_1 y_1 + N_2 y_2 + N_3 y_3 \\ &= \xi y_1 + \eta y_2 + (1 - \xi - \eta) y_3 \\ &= (y_1 - y_3)\xi + (y_2 - y_3)\eta + y_3 \end{aligned}$$

$$\therefore \quad y = y_{13}\xi + y_{23}\eta + y_3$$



Figure

$$x = N_1x_1 + N_2x_2 + N_3x_3$$
$$y = N_1y_1 + N_2y_2 + N_3y_3$$

$$x = x_{13}\xi + x_{23}\eta + x_3$$
$$y = y_{13}\xi + y_{23}\eta + y_3$$

Example 6.1

Evaluate the shape functions N_1, N_2, and N_3 at the interior point P for the triangular element shown in Fig. E6.1.

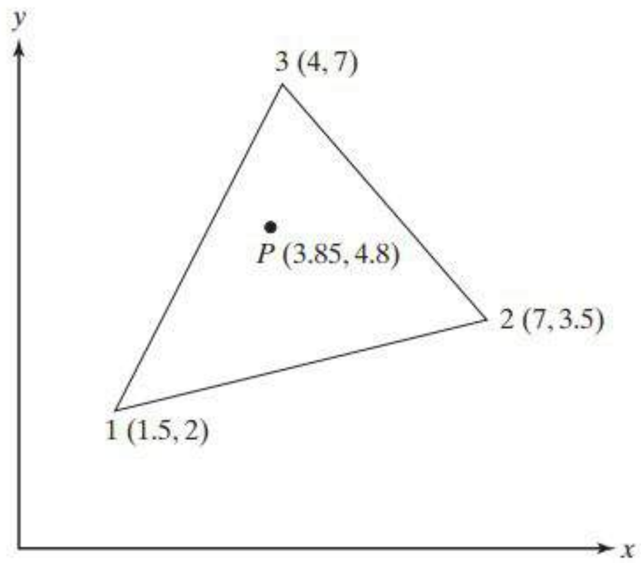
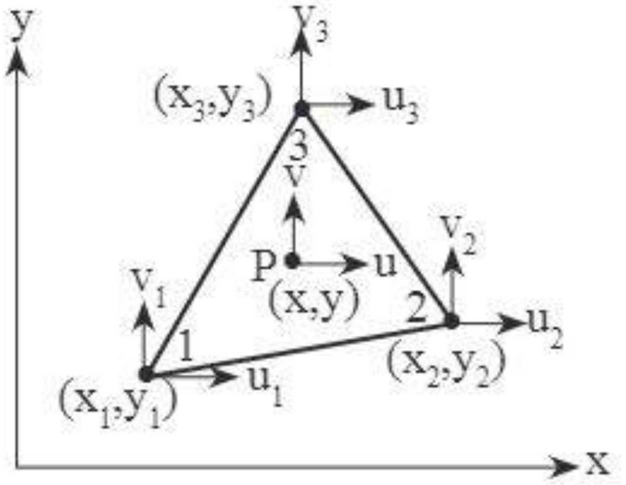


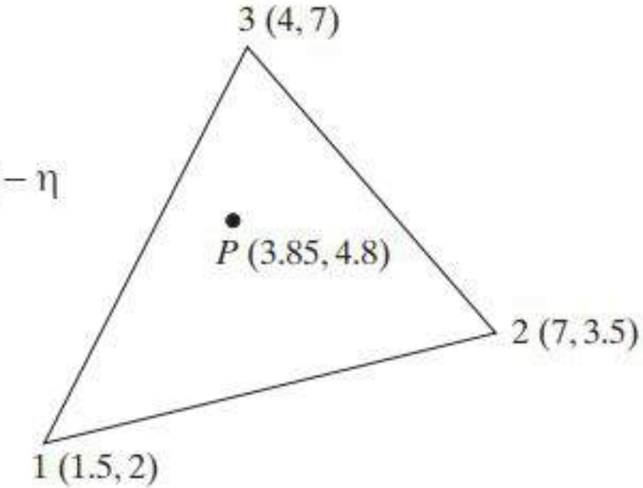
FIGURE E6.1 Examples 6.1 and 6.2.



Figure

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3$$
$$y = N_1 y_1 + N_2 y_2 + N_3 y_3$$

$$N_1 = \xi$$
$$N_2 = \eta$$
$$N_3 = 1 - \xi - \eta$$



$$\boxed{\begin{aligned} 3.85 &= 1.5N_1 + 7N_2 + 4N_3 \\ 4.8 &= 2N_1 + 3.5N_2 + 7N_3 \end{aligned}}$$

$$3.85 = -2.5\xi + 3\eta + 4$$
$$4.8 = -5\xi - 3.5\eta + 7$$

$$2.5\xi - 3\eta = 0.15$$
$$5\xi + 3.5\eta = 2.2$$

Solving the equations, we obtain $\xi = 0.3$ and $\eta = 0.2$,

$$N_1 = 0.3 \qquad N_2 = 0.2 \qquad N_3 = 0.5$$

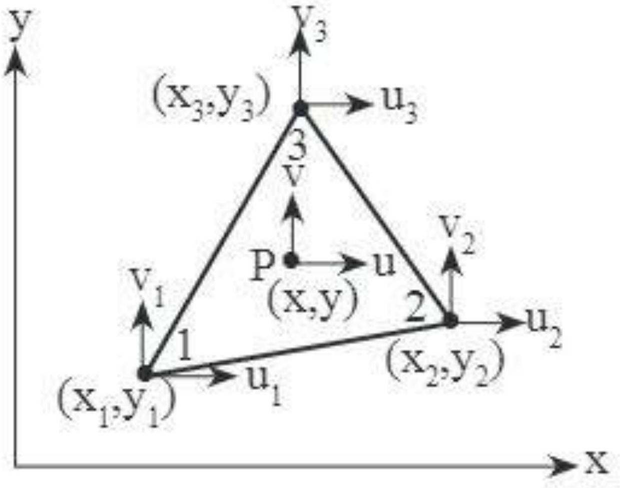
Q. Estimate the shape functions of a triangular element at the point P(22, 44) of a CST with the coordinates 1(0, 0), 2(46, 8) and 3(18, 62). All dimensions are in mm.

Ans: N1=0.11, N2=0.21, N3=0.68

The displacements can be written as,

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3$$

$$v = N_1 v_1 + N_2 v_2 + N_3 v_3$$



Figure

Similarly we can write the coordinates:

$$x = N_1x_1 + N_2x_2 + N_3x_3$$
$$y = N_1y_1 + N_2y_2 + N_3y_3$$

$$N_1 = \xi$$
$$N_2 = \eta$$
$$N_3 = 1 - \xi - \eta$$

$$x = x_{13}\xi + x_{23}\eta + x_3$$
$$y = y_{13}\xi + y_{23}\eta + y_3$$

similarly

$$\mathbf{u} = \mathbf{u}_{13}\xi + \mathbf{u}_{23}\eta + \mathbf{u}_3$$
$$\mathbf{v} = \mathbf{v}_{13}\xi + \mathbf{v}_{23}\eta + \mathbf{v}_3$$

(Recall the strains definitions)

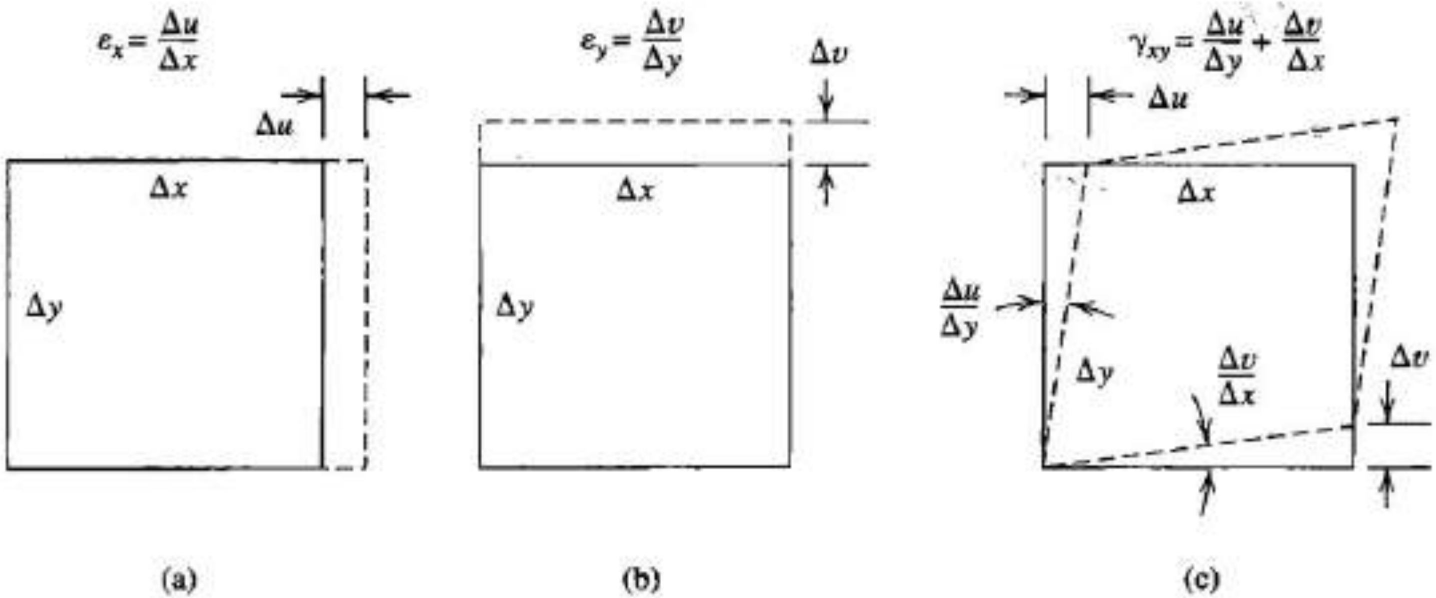


Figure 3.1-1. An infinitesimal rectangle, subjected to (a) *x*-direction normal strain, (b) *y*-direction normal strain, and (c) shear strain.

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \varepsilon_y = \frac{\partial v}{\partial y} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

:In fluid mechanics, velocity V=u i + v j + w k

The velocity of a fluid is not only a function of time but also of space:

$$u = f(x,y,z,t)$$

By the chain rule of differentiation,

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial t}$$

Using the **chain rule** for partial derivatives of u, (u is the x- displacement of nodes)

$$\frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \xi}$$

$$\frac{\partial u}{\partial \eta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \eta}$$

$$\frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \xi}$$

$$\frac{\partial u}{\partial \eta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \eta}$$

which can be written in matrix notation as

$$\begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{Bmatrix}$$

J

(2 * 2) square matrix is denoted as the Jacobian of the transformation, J:

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} x_{13} & y_{13} \\ x_{23} & y_{23} \end{bmatrix}$$

$$\boxed{\begin{aligned} x &= x_{13}\xi + x_{23}\eta + x_3 \\ y &= y_{13}\xi + y_{23}\eta + y_3 \end{aligned}}$$

$$\begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{Bmatrix} \Rightarrow \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{Bmatrix} = \mathbf{J}^{-1} \begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \end{Bmatrix}$$

$$\mathbf{J} = \begin{bmatrix} x_{13} & y_{13} \\ x_{23} & y_{23} \end{bmatrix}$$

where $\mathbf{J}^{-1}$ is the inverse of the Jacobian $\mathbf{J}$, given by

$$\mathbf{J}^{-1} = \frac{1}{\det \mathbf{J}} \begin{bmatrix} y_{23} & -y_{13} \\ -x_{23} & x_{13} \end{bmatrix}$$

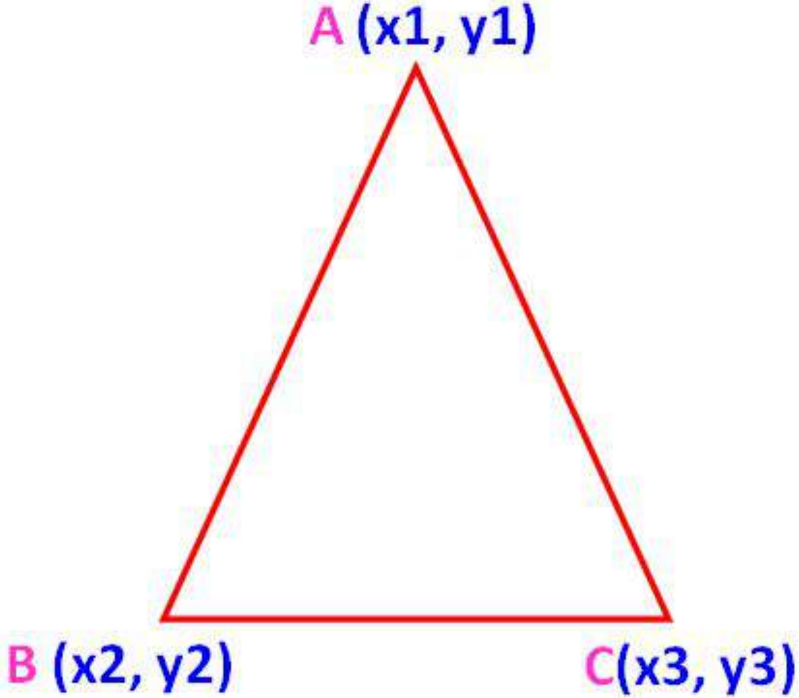
$$\det \mathbf{J} = x_{13}\, y_{23} - x_{23}\, y_{13}$$

From the knowledge of the area of the triangle, it can be seen that the magnitude of det J is twice the area of the triangle

$$A = \tfrac{1}{2}|\det \mathbf{J}|$$

When Matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$



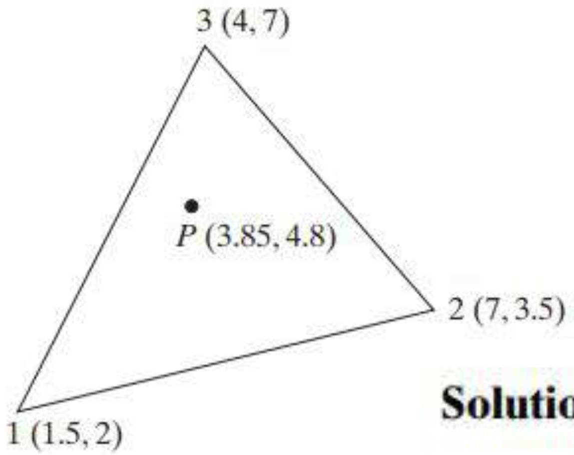
Area of triangle,

$$A = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

$$\boxed{\text{Area of } \Delta = \frac{1}{2}\left\{x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)\right\}}$$

Example 6.2

Determine the Jacobian of the transformation $\mathbf{J}$ for the triangular element shown in Fig. E6.1.



Solution We have

$$\mathbf{J} = \begin{bmatrix} x_{13} & y_{13} \\ x_{23} & y_{23} \end{bmatrix} = \begin{bmatrix} -2.5 & -5.0 \\ 3.0 & -3.5 \end{bmatrix}$$

Thus, $\det \mathbf{J} = 23.75$ units. This is twice the area of the triangle.

$$\begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{Bmatrix} = \mathbf{J}^{-1} \begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \end{Bmatrix}$$

$$\boxed{\begin{aligned} \mathbf{J}^{-1} &= \frac{1}{\det \mathbf{J}} \begin{bmatrix} y_{23} & -y_{13} \\ -x_{23} & x_{13} \end{bmatrix} \\ \det \mathbf{J} &= x_{13}\, y_{23} - x_{23}\, y_{13} \end{aligned}}$$

$$\begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{Bmatrix} = \frac{1}{\det \mathbf{J}} \begin{Bmatrix} y_{23} \dfrac{\partial u}{\partial \xi} - y_{13} \dfrac{\partial u}{\partial \eta} \\ -x_{23} \dfrac{\partial u}{\partial \xi} + x_{13} \dfrac{\partial u}{\partial \eta} \end{Bmatrix}$$

Replacing u by the displacement v, we get a similar expression

$$\begin{Bmatrix} \dfrac{\partial v}{\partial x} \\ \dfrac{\partial v}{\partial y} \end{Bmatrix} = \frac{1}{\det \mathbf{J}} \begin{Bmatrix} y_{23} \dfrac{\partial v}{\partial \xi} - y_{13} \dfrac{\partial v}{\partial \eta} \\ -x_{23} \dfrac{\partial v}{\partial \xi} + x_{13} \dfrac{\partial v}{\partial \eta} \end{Bmatrix}$$

$$\boxed{\begin{aligned} \mathbf{u} &= \mathbf{u}_{13}\xi + \mathbf{u}_{23}\eta + \mathbf{u}_3 \\ \mathbf{v} &= \mathbf{v}_{13}\xi + \mathbf{v}_{23}\eta + \mathbf{v}_3 \end{aligned}}$$

Using the strain–displacement relations

$$\boldsymbol{\epsilon} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix}$$

$$= \frac{1}{\det \mathbf{J}} \begin{Bmatrix} y_{23}(\text{u1 - u3}) - y_{13}(\text{u2 - u3}) \\ -x_{23}(\text{v1 - v3}) + x_{13}(\text{v2 - v3}) \\ -x_{23}(\text{u1 - u3}) + x_{13}(\text{u2 - u3}) + y_{23}(\text{v1 - v3}) - y_{13}(\text{v2 - v3}) \end{Bmatrix}$$

we can write $y_{31} = -y_{13}$ and $y_{12} = y_{13} - y_{23}$

$$\boldsymbol{\epsilon} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix} = \frac{1}{\det \mathbf{J}} \begin{Bmatrix} y_{23}\mathbf{u1} + y_{31}\mathbf{u2} + y_{12}\mathbf{u3} \\ x_{32}\mathbf{v1} + x_{13}\mathbf{v2} + x_{21}\mathbf{v3} \\ x_{32}\mathbf{u1} + y_{23}\mathbf{v1} + x_{13}\mathbf{u2} + y_{31}\mathbf{v2} + x_{21}\mathbf{u3} + y_{12}\mathbf{v3} \end{Bmatrix}$$

$$\boldsymbol{\epsilon} = \mathbf{B}\mathbf{q} \quad = \mathbf{B\ U}$$

$$\mathbf{U} = \begin{Bmatrix} \mathrm{u1} \\ \mathrm{v1} \\ \mathrm{u2} \\ \mathrm{v2} \\ \mathrm{u3} \\ \mathrm{v3} \end{Bmatrix}$$

$$\mathbf{B} = \frac{1}{\det \mathbf{J}} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

First row of B matrix

u y	**u y**	**u y**
123	**2**31	**3**12

where **B** is a (3×6) element strain–displacement matrix relating the three strains to the six nodal displacements and is given by

$$\boldsymbol{\sigma} = \mathbf{D}\,\boldsymbol{\epsilon}$$

Example 6.3

Find the strain–nodal displacement matrices $\mathbf{B}^e$ for the elements shown in Fig. E6.3. Use local numbers given at the corners.

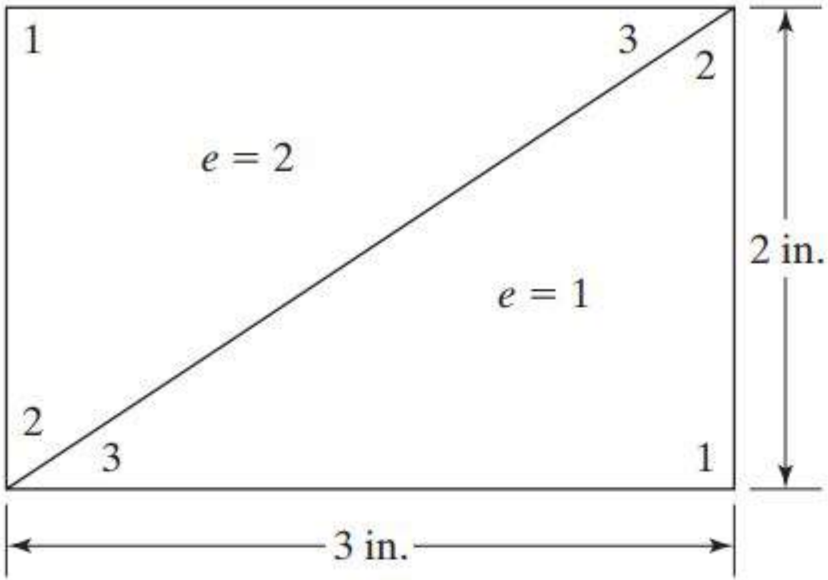


FIGURE E6.3

Solution We have

Remember,

y23= y2 - y3,

x13=x1-x3

$$\mathbf{B}^1 = \frac{1}{\det \mathbf{J}} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

Taking origin at 1

x1,y1=0,0

x2,y2=0,2

x3,y3=-3,0

det $\mathbf{J}$ is obtained from $x_{13}y_{23} - x_{23}y_{13} = (3)(2) - (3)(0) = 6.$

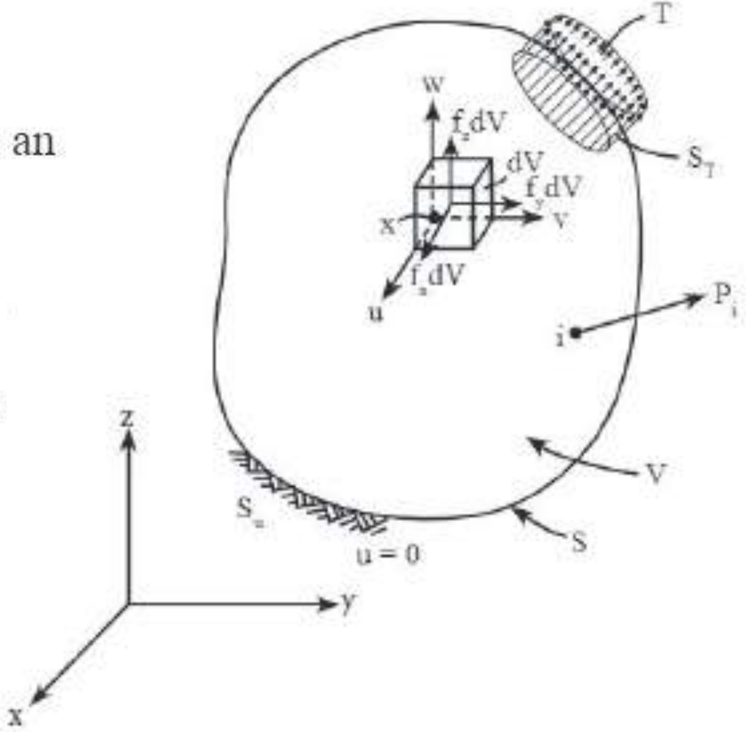
$$= \frac{1}{6} \begin{bmatrix} 2 & 0 & 0 & 0 & -2 & 0 \\ 0 & -3 & 0 & 3 & 0 & 0 \\ -3 & 2 & 3 & 0 & 0 & -2 \end{bmatrix}$$

$$\mathbf{B}^2 = \frac{1}{6} \begin{bmatrix} -2 & 0 & 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & -3 & 0 & 0 \\ 3 & -2 & -3 & 0 & 0 & 2 \end{bmatrix}$$

Potential Energy Approach

General expression for total potential energy in an elastically loaded structure is,

$$\pi = \frac{1}{2}\int_V \sigma^T \varepsilon dV - \int_V u^T f dv - \int_A u^T \mathbf{T} dA - \sum_i u_i^T P_i$$



TPE (π) = Strain Energy + External Work done

$$\boldsymbol{\sigma} = \mathbf{D}\,\boldsymbol{\epsilon}$$

For plane problems,

$$\Pi = \frac{1}{2}\int_A \boldsymbol{\epsilon}^T \mathbf{D} \boldsymbol{\epsilon} t\, dA - \int_A \mathbf{u}^T \mathbf{f} t\, dA - \int_L \mathbf{u}^T \mathbf{T} t\, d\ell - \sum_i \mathbf{u}_i^T \mathbf{P}_i$$

$$\Pi = \frac{1}{2}\int_A \boldsymbol{\epsilon}^{\mathrm{T}}\mathbf{D}\boldsymbol{\epsilon}\, t\, dA - \int_A \mathbf{u}^{\mathrm{T}}\mathbf{f}\, t\, dA - \int_L \mathbf{u}^{\mathrm{T}}\mathbf{T}\, t\, d\ell - \sum_i \mathbf{u}_i^{\mathrm{T}}\mathbf{P}_i$$

$$\Pi = \sum_e \frac{1}{2}\int_e \boldsymbol{\epsilon}^{\mathrm{T}}\mathbf{D}\boldsymbol{\epsilon}\, t\, dA - \sum_e \int_e \mathbf{u}^{\mathrm{T}}\mathbf{f}\, t\, dA - \int_L \mathbf{u}^{\mathrm{T}}\mathbf{T}\, t\, d\ell - \sum_i \mathbf{u}_i^{\mathrm{T}}\mathbf{P}_i$$

$$\Pi = \sum_e U_e - \sum_e \int_e \mathbf{u}^{\mathrm{T}}\mathbf{f}\, t\, dA - \int_L \mathbf{u}^{\mathrm{T}}\mathbf{T}\, t\, d\ell - \sum_i \mathbf{u}_i^{T}\mathbf{P}_i$$

where $U_e = \frac{1}{2}\int_e \boldsymbol{\epsilon}^{\mathrm{T}}\mathbf{D}\boldsymbol{\epsilon}\, t\, dA$ is the element strain energy.

$$U_e = \frac{1}{2}\int_e \boldsymbol{\epsilon}^{\mathrm{T}}\mathbf{D}\boldsymbol{\epsilon}\, t\, dA \qquad \boldsymbol{\epsilon} = \mathbf{B}\mathbf{q} \quad = \mathbf{B}\ \mathbf{U}$$

$$= \frac{1}{2}\int_e \mathbf{q}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B}\mathbf{q}\, t\, dA$$

Element Stiffness, Ke

$$U_e = \frac{1}{2}\int_e \boldsymbol{\epsilon}^{\mathrm{T}}\mathbf{D}\boldsymbol{\epsilon}\, t\, dA$$

$$= \frac{1}{2}\int_e \mathbf{q}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B}\mathbf{q}\, t\, dA$$

Taking the element thickness (t_e) as constant over the element and remembering that all terms in the **D** and **B** matrices are constants, we get

$$U_e = \frac{1}{2}\mathbf{q}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B}t_e\left(\int_e dA\right)\mathbf{q}$$

$\int_e dA = A_e$, where A_e is the area of the element.

$$U_e = \frac{1}{2}\mathbf{q}^{\mathrm{T}}t_e A_e \mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B}\mathbf{q}$$

$$U_e = \frac{1}{2}\mathbf{q}^{\mathrm{T}}\mathbf{k}^e\mathbf{q}$$

where $\mathbf{k}^e$ is the element stiffness matrix given by

$$\mathbf{k}^e = t_e A_e \mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B}$$

$$U_e = \frac{1}{2}\mathbf{q}^{\mathrm{T}}\mathbf{k}^e\mathbf{q}$$

$$U = \sum_e \frac{1}{2}\mathbf{q}^{\mathrm{T}}\mathbf{k}^e\mathbf{q} = \frac{1}{2}\mathbf{Q}^{\mathrm{T}}\mathbf{K}\mathbf{Q} = \tfrac{1}{2}\, U^T K\, U$$

Force Terms

The **body force term** $\int_e \mathbf{u}^{\mathrm{T}}\mathbf{f}t\,dA$

$$\int_e \mathbf{u}^{\mathrm{T}}\mathbf{f}t\,dA = t_e \int_e (uf_x + vf_y)\,dA$$

Using the interpolation relations

Substitute,

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3$$

$$v = N_1 v_1 + N_2 v_2 + N_3 v_3$$

$$\int_e \mathbf{u}^{\mathrm{T}}\mathbf{f}t\,dA = \mathbf{u1}\left(t_e f_x \int_e N_1 dA\right) + \mathbf{v1}\left(t_e f_y \int_e N_1\,dA\right)$$

$$+\mathbf{u2}\left(t_e f_x \int_e N_2\,dA\right) + \mathbf{V2}\left(t_e f_y \int_e N_2\,dA\right)$$

$$+\mathbf{u3}\left(t_e f_x \int_e N_3\,dA\right) + \mathbf{V3}\left(t_e f_y \int_e N_3\,dA\right)$$

$$\int_e N_i\,dA = \frac{1}{3}A_e$$

$$\int_e N_1\,dA = \frac{1}{3}A_e h = \frac{1}{3}A_e$$

$$\int_e \mathbf{u}^{\mathrm{T}}\mathbf{f}t\,dA = \mathbf{q}^{\mathrm{T}}\mathbf{f}^e$$

$$\int_e N_2\,dA = \int_e N_3\,dA = 1/3\,A_e.$$

$$\mathbf{f}^e = \frac{t_e A_e}{3}[f_x \quad f_y \quad f_x \quad f_y \quad f_x \quad f_y]^{\mathrm{T}}$$

Traction force vector

$$\mathbf{T}^e = \frac{t_e \ell_{1-2}}{6}[2T_{x_1} + T_{x_2} \quad 2T_{y_1} + T_{y_2} \quad T_{x_1} + 2T_{x_2} \quad T_{y_1} + 2T_{y_2}]^{\mathrm{T}}$$

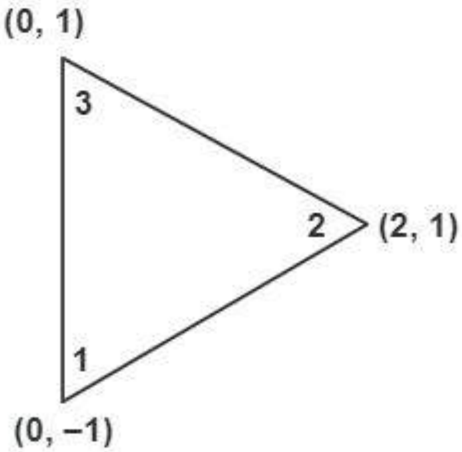
$$\boxed{\Pi = \frac{1}{2}\int_A \boldsymbol{\epsilon}^{\mathrm{T}}\mathbf{D}\boldsymbol{\epsilon}\, t\, dA - \int_A \mathbf{u}^{\mathrm{T}}\mathbf{f}\, t\, dA - \int_L \mathbf{u}^{\mathrm{T}}\mathbf{T}\, t\, d\ell - \sum_i \mathbf{u}_i^{\mathrm{T}}\mathbf{P}_i}$$

$$\Downarrow$$

$$\boxed{\Pi = \frac{1}{2}\mathbf{Q}^{\mathrm{T}}\mathbf{K}\mathbf{Q} - \mathbf{Q}^{\mathrm{T}}\mathbf{f}}$$

$\mathbf{KQ} = \mathbf{F}$ **(Or) K U = F**

Q. Assuming plane stress condition, evaluate stiffness matrix for the element shown in figure. Assume E = 200 GPa, Poisson's ratio 0.3.



Figure

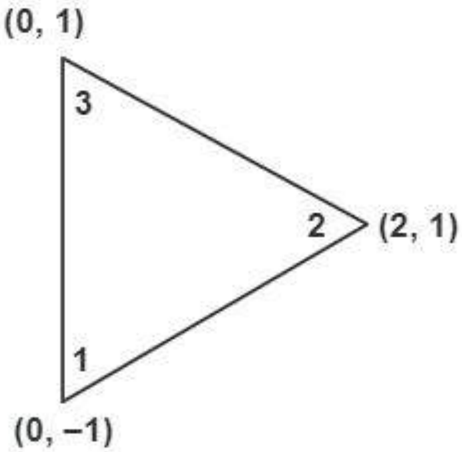
Coordinates are in mm

Q. Assuming plane stress condition, evaluate stiffness matrix for the element shown in figure. Assume E = 200 GPa, Poisson's ratio 0.3.

$$A = \tfrac{1}{2}|\det \mathbf{J}|$$

$$\boxed{\mathbf{J} = \begin{bmatrix} x_{13} & y_{13} \\ x_{23} & y_{23} \end{bmatrix}}$$

$$\mathbf{B} = \frac{1}{\det \mathbf{J}} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$



Figure

Coordinates are in mm

Plane stress: $\mathbf{D} = \dfrac{E}{1-\nu^2}\begin{pmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{pmatrix}$

Plane strain: $\mathbf{D} = \dfrac{E}{(1+\nu)(1-2\nu)}\begin{pmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{pmatrix}$

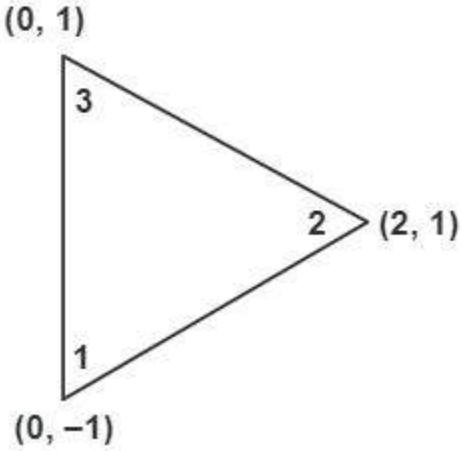
$\mathbf{k}^e$ is the element stiffness matrix given by

$$\mathbf{k}^e = t_e A_e \mathbf{B}^{\mathrm{T}} \mathbf{D} \mathbf{B}$$

Given that,

Young's modulus, $E = 200$ GPa $= 200 \times 10^3$ N/mm^2

Poisson's ratio, $\mu = 0.3$



Global coordinates of ,

Node-1, $(x_1, y_1) = (0, -1)$

Node-2, $(x_2, y_2) = (2, 0)$

Node-3, $(x_3, y_3) = (0, 1)$

Assume,

Thickness, $t = 1$ mm

And, given coordinate are in mm.

Stiffness matrix for linear triangular element,

$$[K] = [B]^T [D] [B] A.t$$

Stress-strain relationship matrix, considering plane stress condition,

$$[D] = \frac{E}{1-\mu^2}\begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix} = \frac{200\times 10^3}{1-0.3^2}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & \frac{1-0.3}{2} \end{bmatrix}$$

$$[D] = 219.78\times 10^3\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}$$

$$\boxed{\mathbf{J} = \begin{bmatrix} x_{13} & y_{13} \\ x_{23} & y_{23} \end{bmatrix}}$$

$$\det \mathbf{J} = x_{13}\, y_{23} - x_{23}\, y_{13}$$

$$A = \tfrac{1}{2}|\det \mathbf{J}|$$

$$\mathbf{B} = \frac{1}{\det \mathbf{J}} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

$$x_3 - x_2 = 0 - 2 = -2$$

$$x_1 - x_3 = 0 - 0 = 0$$

$$x_2 - x_1 = 2 \times 0 = 2$$

$$y_2 - y_3 = 0 - 1 = -1$$

$$y_3 - y_1 = 1 - (-1) = 2$$

$$y_1 - y_2 = (-1) - 0 = -1$$

$$[B] = \frac{1}{4}\begin{bmatrix} -1 & 0 & 2 & 0 & -1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2 \\ -2 & -1 & 0 & 2 & 2 & -1 \end{bmatrix} \qquad [B]^T = \frac{1}{4}\begin{bmatrix} -1 & 0 & -2 \\ 0 & -2 & -1 \\ 2 & 0 & 0 \\ 0 & 0 & 2 \\ -1 & 0 & 2 \\ 0 & 2 & -1 \end{bmatrix}$$

Stiffness matrix for linear triangular element,

$$[K] = [B]^T\,[D]\,[B]\,A.t$$

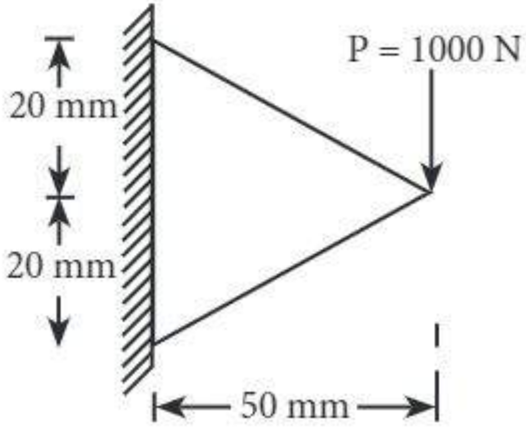
$$[K] = \frac{1}{4}\begin{bmatrix} -1 & 0 & -2 \\ 0 & -2 & -1 \\ 2 & 0 & 0 \\ 0 & 0 & 2 \\ -1 & 0 & 2 \\ 0 & 2 & -1 \end{bmatrix} \times 219.78 \times 10^3 \begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix} \times \frac{1}{4}\begin{bmatrix} -1 & 0 & 2 & 0 & -1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2 \\ -2 & -1 & 0 & 2 & 2 & -1 \end{bmatrix} \times 2 \times 1$$

$$= 27472.5 \begin{bmatrix} -1 & 0 & -2 \\ 0 & -2 & -1 \\ 2 & 0 & 0 \\ 0 & 0 & 2 \\ -1 & 0 & 2 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix} \begin{bmatrix} -1 & 0 & 2 & 0 & -1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2 \\ -2 & -1 & 0 & 2 & 2 & -1 \end{bmatrix}$$

$$= 27472.5 \begin{bmatrix} -1 & -0.3 & -0.7 \\ -0.6 & -2 & -0.35 \\ 2 & 0.6 & 0 \\ 0 & 0 & 0.7 \\ -1 & -0.3 & 0.7 \\ 0.6 & 2 & -0.35 \end{bmatrix} \begin{bmatrix} -1 & 0 & 2 & 0 & -1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2 \\ -2 & -1 & 0 & 2 & 2 & -1 \end{bmatrix}$$

$$[K] = 27472.5 \begin{bmatrix} 2.4 & 1.3 & -2 & -1.4 & -0.4 & 0.1 \\ 1.3 & 4.35 & -1.2 & -0.7 & -0.1 & -3.65 \\ -2 & -1.2 & 4 & 0 & -2 & 1.2 \\ -1.4 & -0.7 & 0 & 1.4 & 1.4 & -0.7 \\ -0.4 & -0.1 & -2 & 1.4 & 2.4 & -1.3 \\ 0.1 & -3.65 & 1.2 & -0.7 & -1.3 & 4.35 \end{bmatrix}$$

Q. It is required to determine the transverse displacement and the stresses induced in the plate shown in figure using a one-element idealization. Determine the constitutive matrix and the strain-displacement matrix and hence the stiffness matrix and the load vector. Assume E = 205 GPa, $\mu = 0.33$, and t = 10 mm.



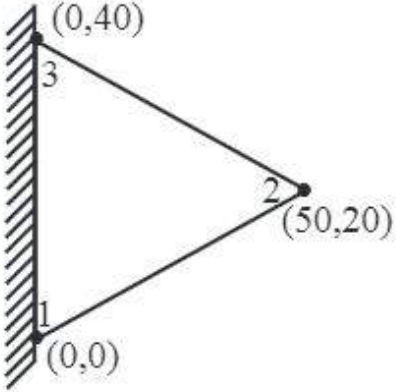
Figure

Given that,

Young's modulus, $E = 205$ GPa $= 205 \times 10^3$ N/mm^2

Poisson's ratio, $\mu = 0.33$

Thickness, $t = 10$ mm



Coordinates of,

Node-1, $(x_1, y_1) = (0, 0)$

Node-2, $(x_2, y_2) = (50, 20)$

Node-3, $(x_3, y_3) = (0, 40)$

Transverse Displacement

Nodal Displacement can be obtained by using the relation,

$$[K]\,\{\delta\} = \{F\}$$

Stiffness matrix for linear triangular element,

$$[K] = [B]^T [D] [B] A.t$$

$$\boxed{\mathbf{J} = \begin{bmatrix} x_{13} & y_{13} \\ x_{23} & y_{23} \end{bmatrix}}$$

Area of triangle,

$$\det \mathbf{J} = x_{13}\,y_{23} - x_{23}\,y_{13}$$

$$A = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

or

$$A = \tfrac{1}{2}\left|\det \mathbf{J}\right|$$

$$= \frac{1}{2}[2000]$$

Constitutive matrix or stress-strain matrix,

$$[D] = \frac{E}{1-\mu^2}\begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix}$$

$$= \frac{205\times10^3}{1-0.33^2}\begin{bmatrix} 1 & 0.33 & 0 \\ 0.33 & 1 & 0 \\ 0 & 0 & \frac{1-0.33}{2} \end{bmatrix}$$

$$\boxed{[D] = 230.052\times10^3\begin{bmatrix} 1 & 0.33 & 0 \\ 0.33 & 1 & 0 \\ 0 & 0 & 0.335 \end{bmatrix}}$$

$$\mathbf{B} = \frac{1}{\det \mathbf{J}} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

$$y_2 - y_3 = 20 - 40 = -20$$

$$y_3 - y_1 = 40 - 0 = 40$$

$$y_1 - y_2 = 0 - 20 = -20$$

$$x_3 - x_2 = 0 - 50 = -50$$

$$x_1 - x_3 = 0 - 0 = 0$$

$$x_2 - x_1 = 50 - 0 = 50$$

Then,

$$[B] = \frac{1}{2 \times 1000} \begin{bmatrix} -20 & 0 & 40 & 0 & -20 & 0 \\ 0 & -50 & 0 & 0 & 0 & 50 \\ -50 & -20 & 0 & 40 & 50 & -20 \end{bmatrix}$$

$$[B] = \frac{1}{2000} \begin{bmatrix} -20 & 0 & 40 & 0 & -20 & 0 \\ 0 & -50 & 0 & 0 & 0 & 50 \\ -50 & -20 & 0 & 40 & 50 & -20 \end{bmatrix}$$

And,

$$[B]^T = \frac{1}{2000} \begin{bmatrix} -20 & 0 & -50 \\ 0 & -50 & -20 \\ 40 & 0 & 0 \\ 0 & 0 & 40 \\ -20 & 0 & 50 \\ 0 & 50 & -20 \end{bmatrix}$$

$$[K] = \frac{1}{2000}\begin{bmatrix} -20 & 0 & -50 \\ 0 & -50 & -50 \\ 40 & 0 & 0 \\ 0 & 0 & 40 \\ -20 & 0 & 50 \\ 0 & 50 & -20 \end{bmatrix} \times 230.052 \times 10^3 \begin{bmatrix} 1 & 0.33 & 0 \\ 0.33 & 1 & 0 \\ 0 & 0 & 0.335 \end{bmatrix}$$

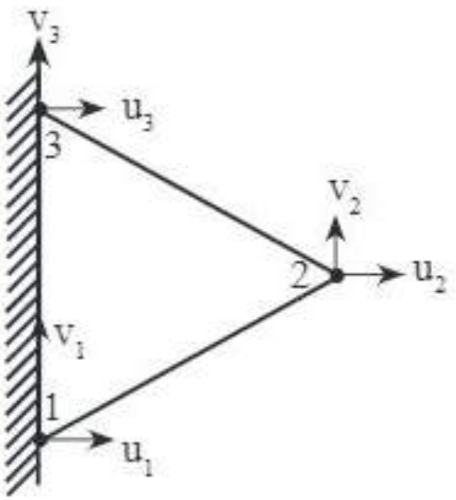
$$\times \frac{1}{2000}\begin{bmatrix} -20 & 0 & 40 & 0 & -20 & 0 \\ 0 & -50 & 0 & 0 & 0 & 50 \\ -50 & -20 & 0 & 40 & 50 & -20 \end{bmatrix} \times 1000 \times 10$$

$$= 575.13\begin{bmatrix} -20 & 0 & -50 \\ 0 & -50 & -20 \\ 40 & 0 & 0 \\ 0 & 0 & 40 \\ -20 & 0 & 50 \\ 0 & 50 & -20 \end{bmatrix}\begin{bmatrix} 1 & 0.33 & 0 \\ 0.33 & 1 & 0 \\ 0 & 0 & 0.335 \end{bmatrix}\begin{bmatrix} -20 & 0 & 40 & 0 & -20 & 0 \\ 0 & -50 & 0 & 0 & 0 & 50 \\ -50 & -20 & 0 & 40 & 50 & -20 \end{bmatrix}$$

$$= 575.13 \begin{bmatrix} -20 & 0 & -50 \\ 0 & -50 & -20 \\ 40 & 0 & 0 \\ 0 & 0 & 40 \\ -20 & 0 & 50 \\ 0 & 50 & -20 \end{bmatrix} \begin{bmatrix} -20 & -16.5 & 40 & 0 & -20 & 16.5 \\ -6.6 & -50 & 13.2 & 0 & -6.6 & 50 \\ -16.75 & -6.7 & 0 & 13.4 & 16.75 & -6.7 \end{bmatrix}$$

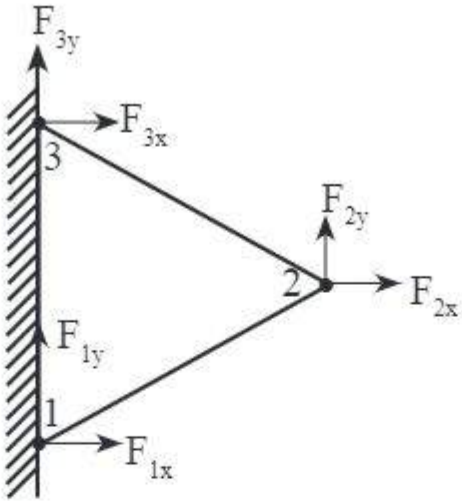
$$[K] = 575.13 \begin{bmatrix} 1237.5 & 665 & -800 & -670 & -437.5 & 5 \\ 665 & 2634 & -660 & -268 & -5 & -2366 \\ -800 & -660 & 1600 & 0 & -800 & 660 \\ -670 & -268 & 0 & 536 & 670 & -268 \\ -437.5 & -5 & -800 & 670 & 1237.5 & -665 \\ 5 & -2366 & 660 & -268 & -665 & -2634 \end{bmatrix}$$

Verification: Evaluated stiffness matrix is said to be correct if it satisfies two conditions such as it should be symmetric and sum of values of any row or column should be zero. As these conditions are satisfied by obtained stiffness matrix, it is said to be correct.



Figure

$$\{\delta\} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$



Figure

$$\{F\} = \begin{Bmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ F_{2y} \\ F_{3x} \\ F_{3y} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -1000 \\ 0 \\ 0 \end{Bmatrix}$$

$$575.13\begin{bmatrix} 1237.5 & 665 & -800 & -670 & -437.5 & 5 \\ 665 & 2634 & -660 & -268 & -5 & -2366 \\ -800 & -660 & 1600 & 0 & -800 & 660 \\ -670 & -268 & 0 & 536 & 670 & -268 \\ -437.5 & -5 & -800 & 670 & 1237.5 & -665 \\ 5 & -2366 & 660 & -268 & -665 & 2364 \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -1000 \\ 0 \\ 0 \end{Bmatrix}$$

Applying boundary conditions,

$$u_1 = v_1 = u_3 = v_3 = 0$$

$$575.13\begin{bmatrix} 1600 & 0 \\ 0 & 536 \end{bmatrix}\begin{Bmatrix} u_2 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -1000 \end{Bmatrix}$$

On writing above matrix in equation form,

$$575.13 \times 1600\, u_2 = 0$$

$$u_2 = 0 \text{ mm}$$

Also,

$$575.13 \times 536\, v_2 = -1000$$

$$v_2 = \frac{-1000}{575.13 \times 536}$$

$$v_2 = -3.244 \times 10^{-3} \text{ mm} \qquad \text{('–ve' sign indicates downward displacement)}$$

Stresses Induced in the plate

Stress vector for linear triangular element,

$$\{\sigma\} = [D]\,[B]\,\{\delta\}$$

$$\{\sigma\} = 230.052 \times 10^3 \begin{bmatrix} 1 & 0.33 & 0 \\ 0.33 & 1 & 0 \\ 0 & 0 & 0.335 \end{bmatrix} \times \frac{1}{2000} \begin{bmatrix} -20 & 0 & 40 & 0 & -20 & 0 \\ 0 & -50 & 0 & 0 & 0 & 50 \\ -50 & -20 & 0 & 40 & 50 & -20 \end{bmatrix} \times 10^{-3} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -3.244 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix}\sigma_x\\\sigma_y\\\tau_{xy}\end{Bmatrix} = 0.115\begin{bmatrix}1 & 0.33 & 0\\0.33 & 1 & 0\\0 & 0 & 0.335\end{bmatrix}\begin{bmatrix}-20 & 0 & 40 & 0 & -20 & 0\\0 & -50 & 0 & 0 & 0 & 50\\-50 & -20 & 0 & 40 & 50 & -20\end{bmatrix}\begin{Bmatrix}0\\0\\0\\-3.244\\0\\0\end{Bmatrix}$$

$$\begin{Bmatrix}\sigma_x\\\sigma_y\\\tau_{xy}\end{Bmatrix} = 0.115\begin{bmatrix}-20 & -16.5 & 40 & 0 & -20 & 16.5\\-6.6 & -50 & 13.2 & 0 & -6.6 & 50\\-16.75 & -6.7 & 0 & 13.4 & 16.75 & -6.7\end{bmatrix}\begin{Bmatrix}0\\0\\0\\-3.244\\0\\0\end{Bmatrix}$$

$$\begin{Bmatrix}\sigma_x\\\sigma_y\\\tau_{xy}\end{Bmatrix} = 0.115\begin{Bmatrix}0\\0\\-43.47\end{Bmatrix}$$

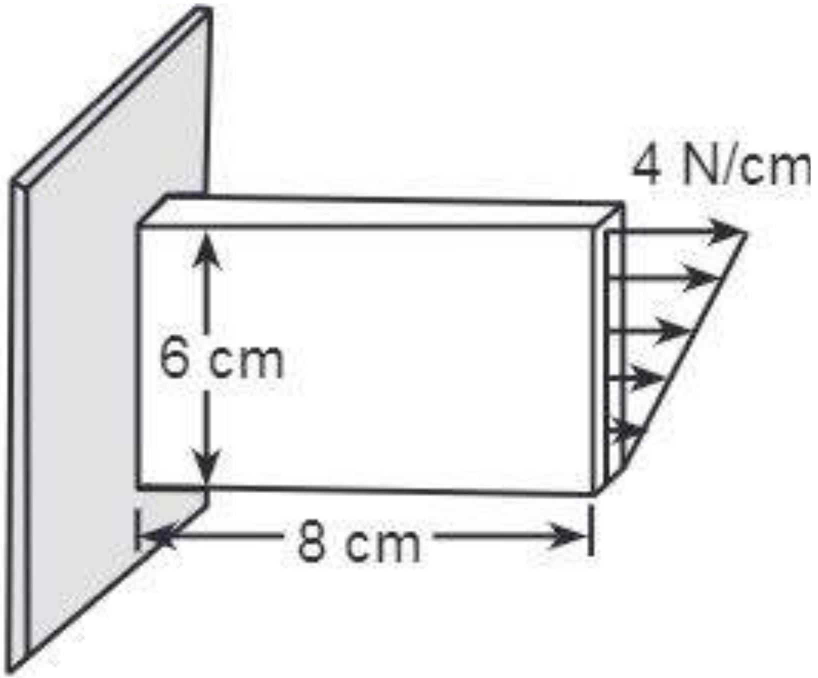
$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -4.999 \end{Bmatrix} \text{N/mm}^2$$

Therefore,

Normal stress in x – direction, $\sigma_x = 0$ N/mm^2

Normal stress in y – direction, $\sigma_y = 0$ N/mm^2

Shear stress in x-y plane, $\tau_{xy} = -4.999$ N/mm^2



Figure

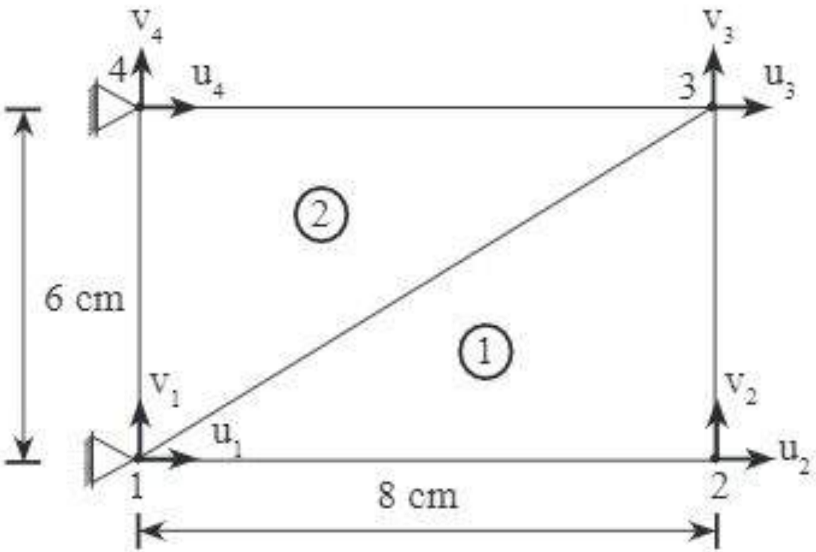
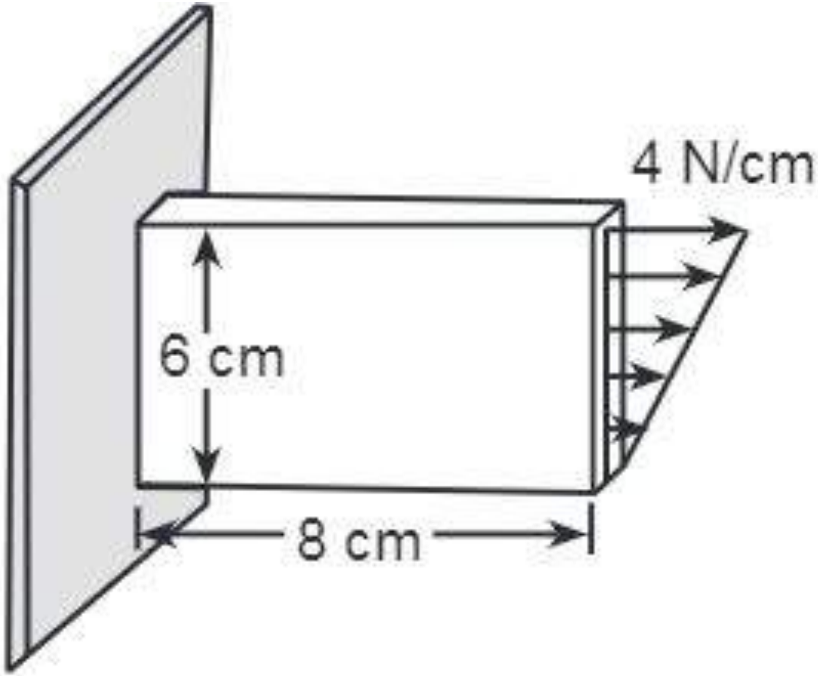
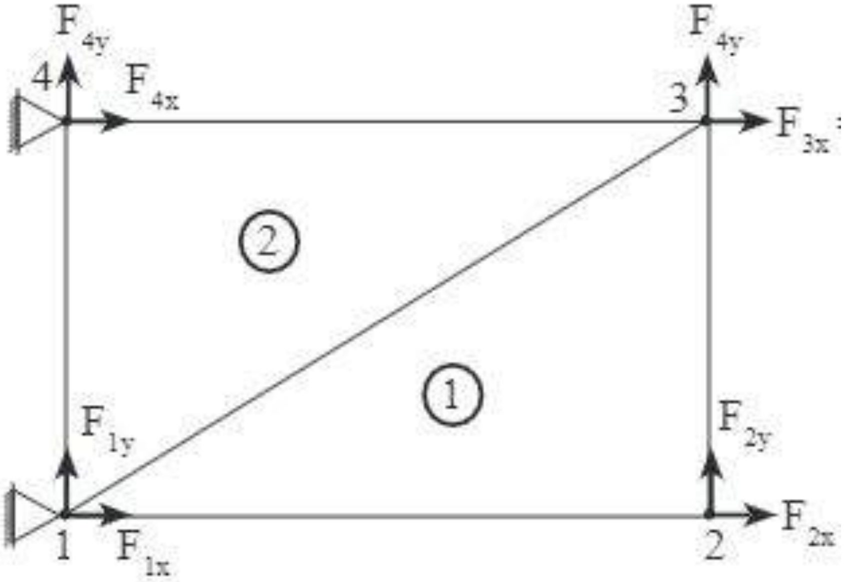
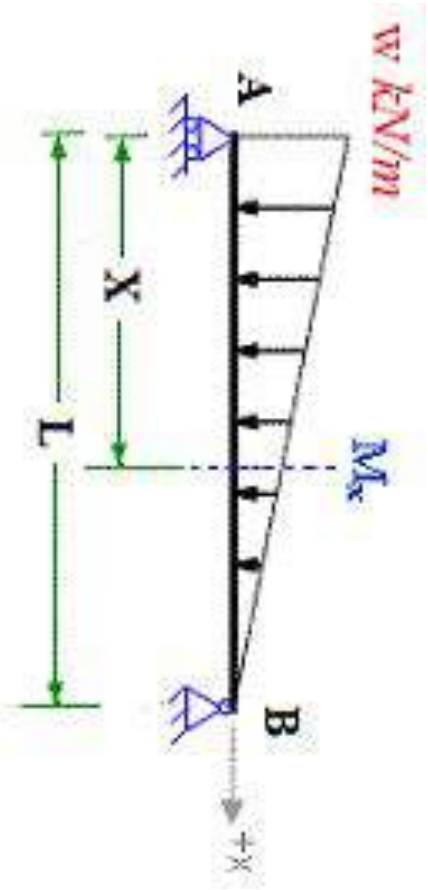


Figure: Nodal Displacements

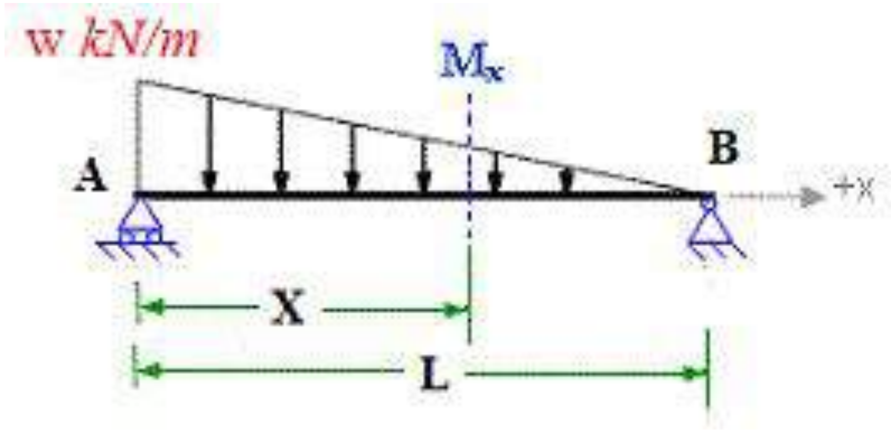


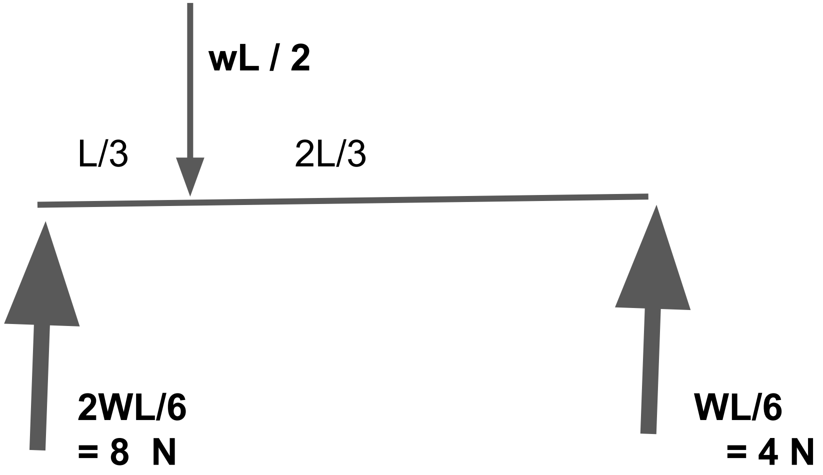


Figure



W = 4
L = 6





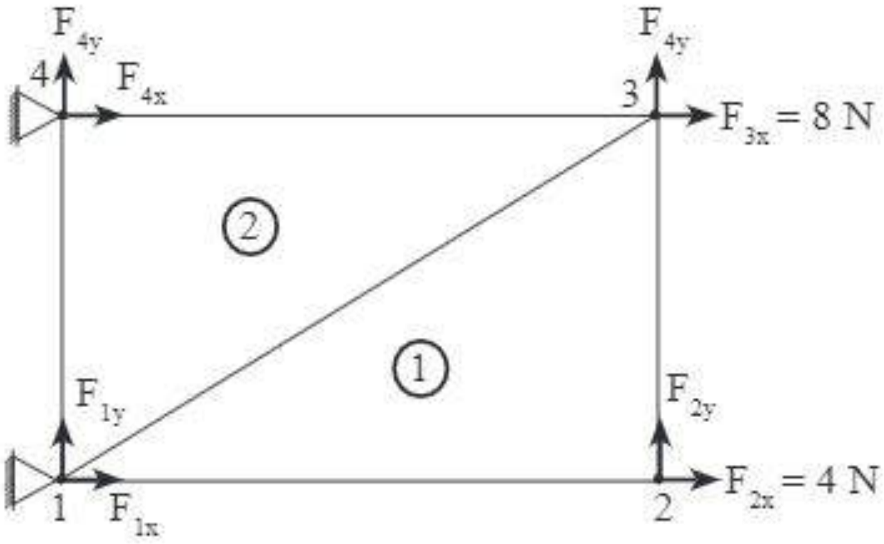
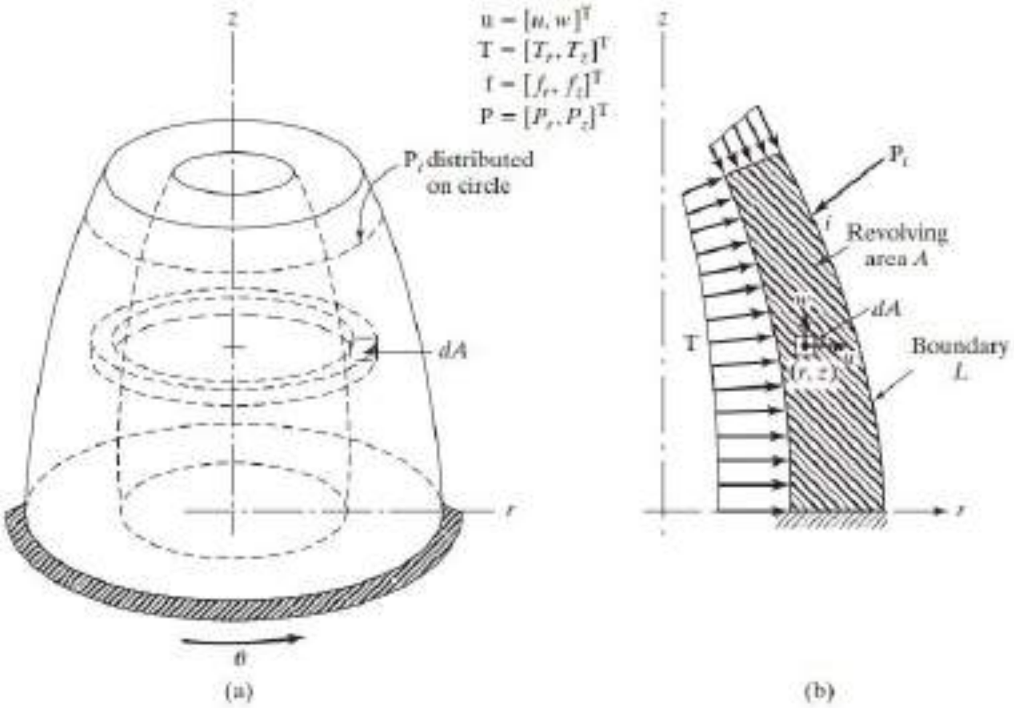


Figure: Nodal Forces

Axisymmetric Solids Subjected to Axisymmetric Loading

The problem is said to be axisymmetric type, if the object has an axis of symmetry and parameters such as geometry boundary conditions, loading and materials are symmetric about this axis. Axisymmetric solids are also known as solids of revolutions. Analysis of such problems is termed as axisymmetric analysis.



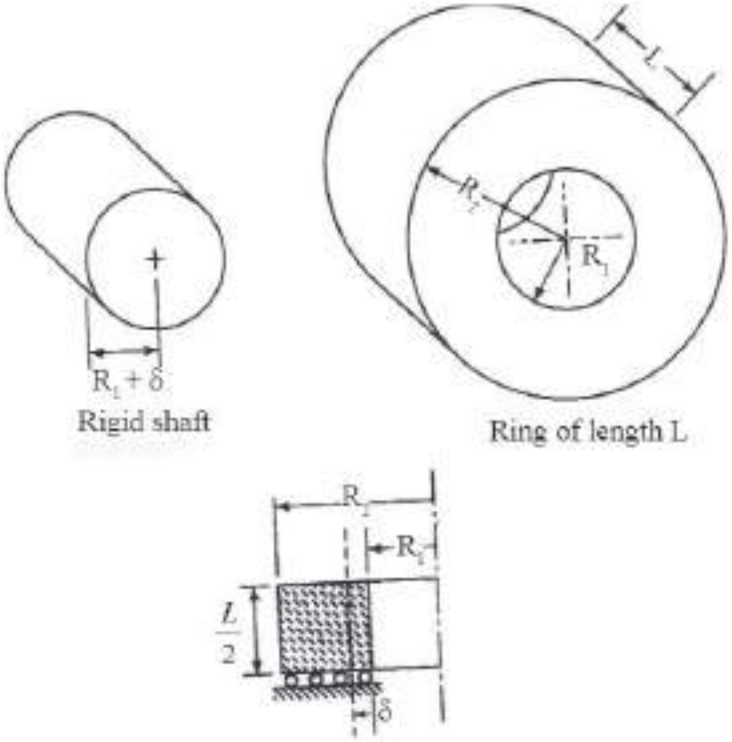


Figure (1): Press Fit of Ring on a Rigid Shaft

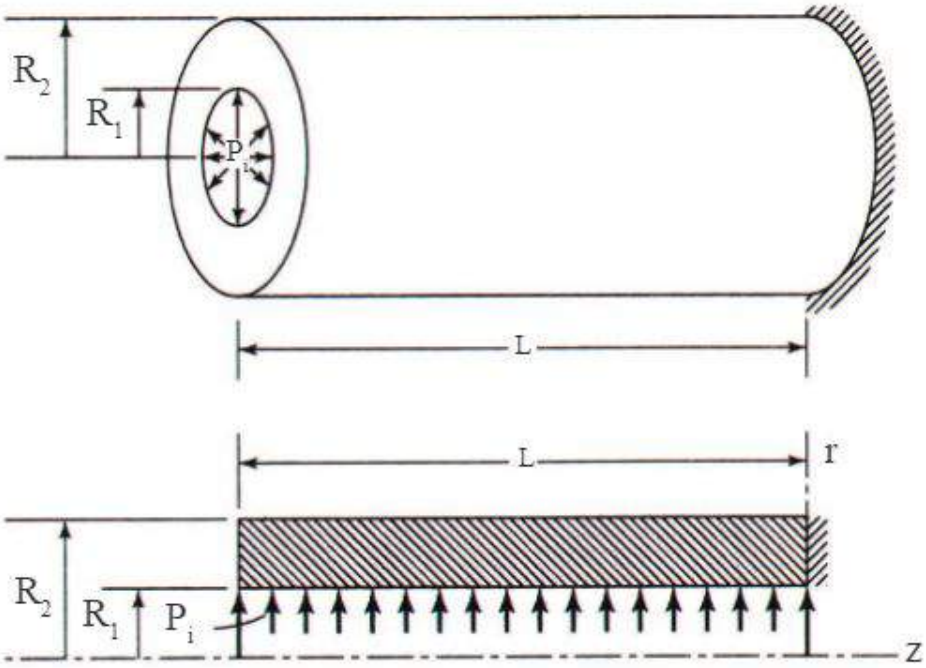
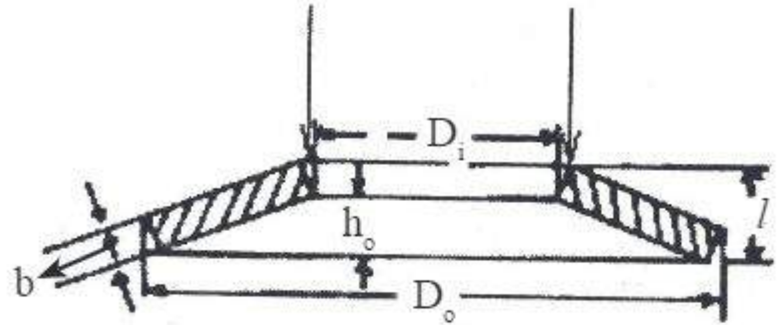
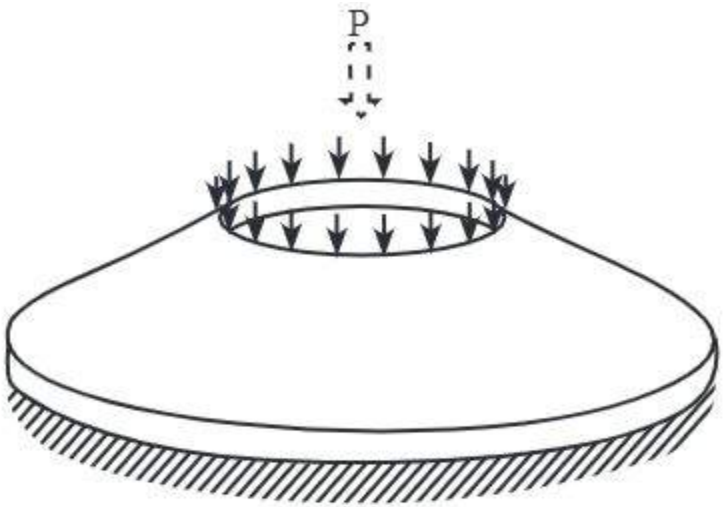


Figure (3): Hollow Cylinder Subjected to Internal Pressure

Belleville (Disk) spring



Figure



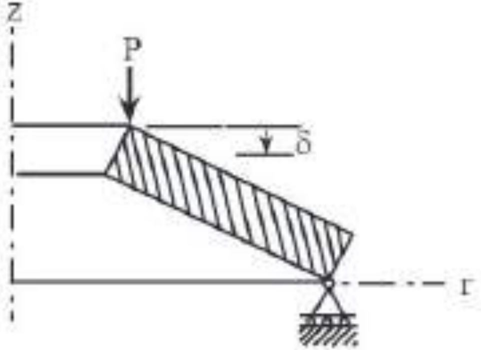


Figure (2)

coordinate system are r, θ, z and u, v, w are their respective displacement functions.

In axisymmetric problem, parameters such as surface loading and geometry are independent of the circumferential direction 'θ'. Thereby, displacement in circumferential direction 'v' will be zero. Thus, only displacements corresponding to direction 'r' and 'z' remains.

$$\text{i.e., } u = f(r, z)$$
$$v = 0$$
$$w = f(r, z)$$

Strain occurring in the element,

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_r \\ \varepsilon_\theta \\ \varepsilon_z \\ \gamma_{rz} \end{Bmatrix}$$

$$\{\varepsilon\} = \begin{Bmatrix} \dfrac{\partial u}{\partial r} \\ \dfrac{u}{r} \\ \dfrac{\partial w}{\partial z} \\ \dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial r} \end{Bmatrix}$$

$$\{\sigma\} = \begin{Bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \\ \tau_{rz} \end{Bmatrix}$$

Where,

σ_r – Radial stress

σ_θ – Circumferential or tangential stress

σ_z – Longitudinal or axial stress

τ_{rz} – Shear stress.

Using Hooke's law, for stress-strain relationship,

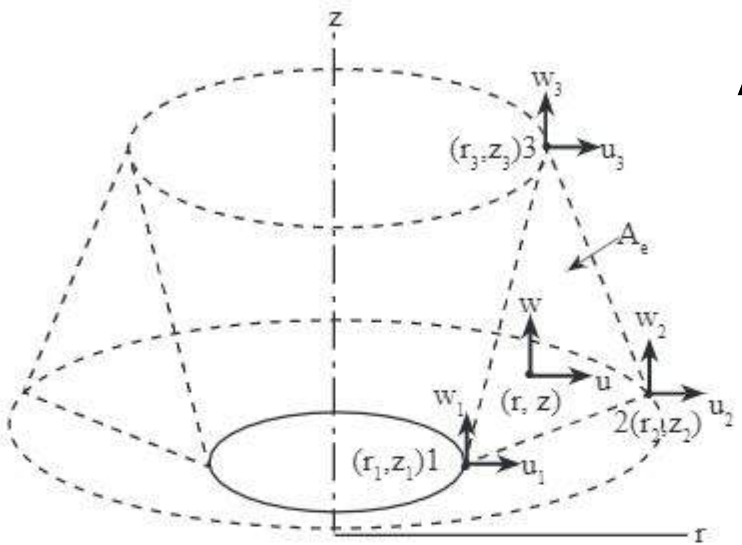
$$\{\sigma\} = [D]\,\{\varepsilon\}$$

Where,

$[D]$ – Stress– strain relationship matrix.

$$\begin{Bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \\ \tau_{rz} \end{Bmatrix} = \frac{E}{(1+\mu)(1-2\mu)} \begin{bmatrix} 1-\mu & \mu & \mu & 0 \\ \mu & 1-\mu & \mu & 0 \\ \mu & \mu & 1-\mu & 0 \\ 0 & 0 & 0 & \dfrac{1-2\mu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_r \\ \varepsilon_\theta \\ \varepsilon_z \\ \gamma_{rz} \end{Bmatrix}$$

$$\{\sigma\} = [D]\{\varepsilon\}$$



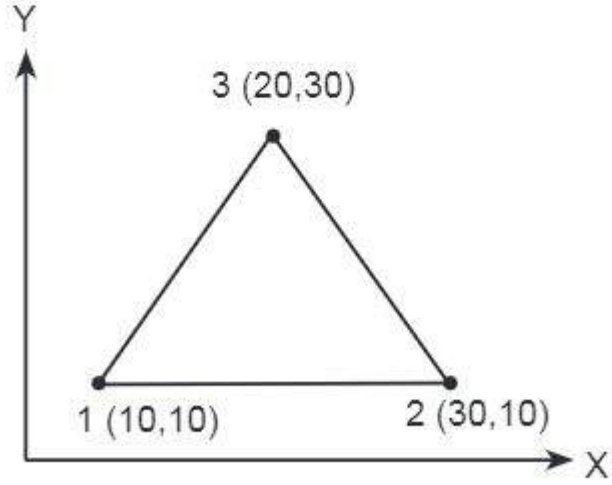
Figure

Axisymmetric element:

$$[D] = \frac{E}{(1+\mu)(1-2\mu)}\begin{bmatrix} 1-\mu & \mu & \mu & 0 \\ \mu & 1-\mu & \mu & 0 \\ \mu & \mu & 1-\mu & 0 \\ 0 & 0 & 0 & \dfrac{1-2\mu}{2} \end{bmatrix}$$

Plane stress: $\mathbf{D} = \frac{E}{1-\nu^2}\begin{pmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{pmatrix}$

Plane strain: $\mathbf{D} = \frac{E}{(1+\nu)(1-2\nu)}\begin{pmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{pmatrix}$



Figure

Q.. Calculate the stiffness matrix for the axisymmetric element Shown in the figure

Given that,

Coordinates of,

Node-1, $(r_1, z_1) = (10, 10)$

Node-2, $(r_2, z_2) = (30, 10)$

Node-3, $(r_3, z_3) = (20, 30)$

Assume, given coordinates are in mm.

Constitutive Matrix

Assume,

Young's modulus, $E = 2 \times 10^5$ N/mm^2

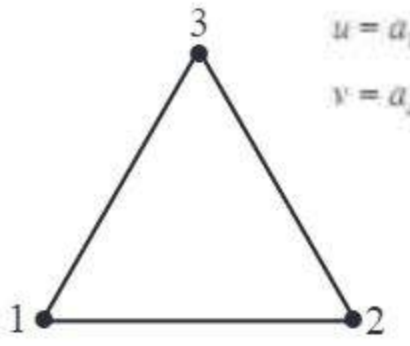
Poisson's ratio, $\mu = 0.3$

Constitutive matrix or stress-strain relationship matrix is given by,

$$[D] = \frac{E}{(1+\mu)(1-2\mu)}\begin{bmatrix} 1-\mu & \mu & \mu & 0 \\ \mu & 1-\mu & \mu & 0 \\ \mu & \mu & 1-\mu & 0 \\ 0 & 0 & 0 & \left(\frac{1-2\mu}{2}\right) \end{bmatrix} = \frac{2 \times 10^5}{(1+0.3)(1-0.6)}\begin{bmatrix} 1-0.3 & 0.3 & 0.3 & 0 \\ 0.3 & 1-0.3 & 0.3 & 0 \\ 0.3 & 0.3 & 1-0.3 & 0 \\ 0 & 0 & 0 & \left(\frac{1-0.6}{2}\right) \end{bmatrix}$$

$$\therefore \quad [D] = 38.4 \times 10^3 \begin{bmatrix} 7 & 3 & 3 & 0 \\ 3 & 7 & 3 & 0 \\ 3 & 3 & 7 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Higher Order elements



$$u = a_1 + a_2 x + a_3 y$$

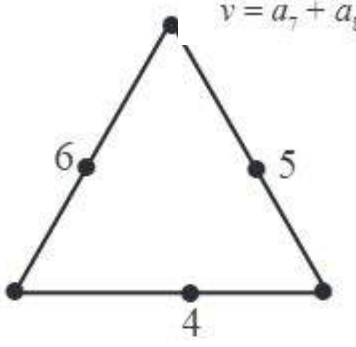
$$v = a_4 + a_5 x + a_6 y$$

(a) Linear Element (n = 1)

Figure(1): Single Order Element

$$u = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 xy + a_6 y^2$$

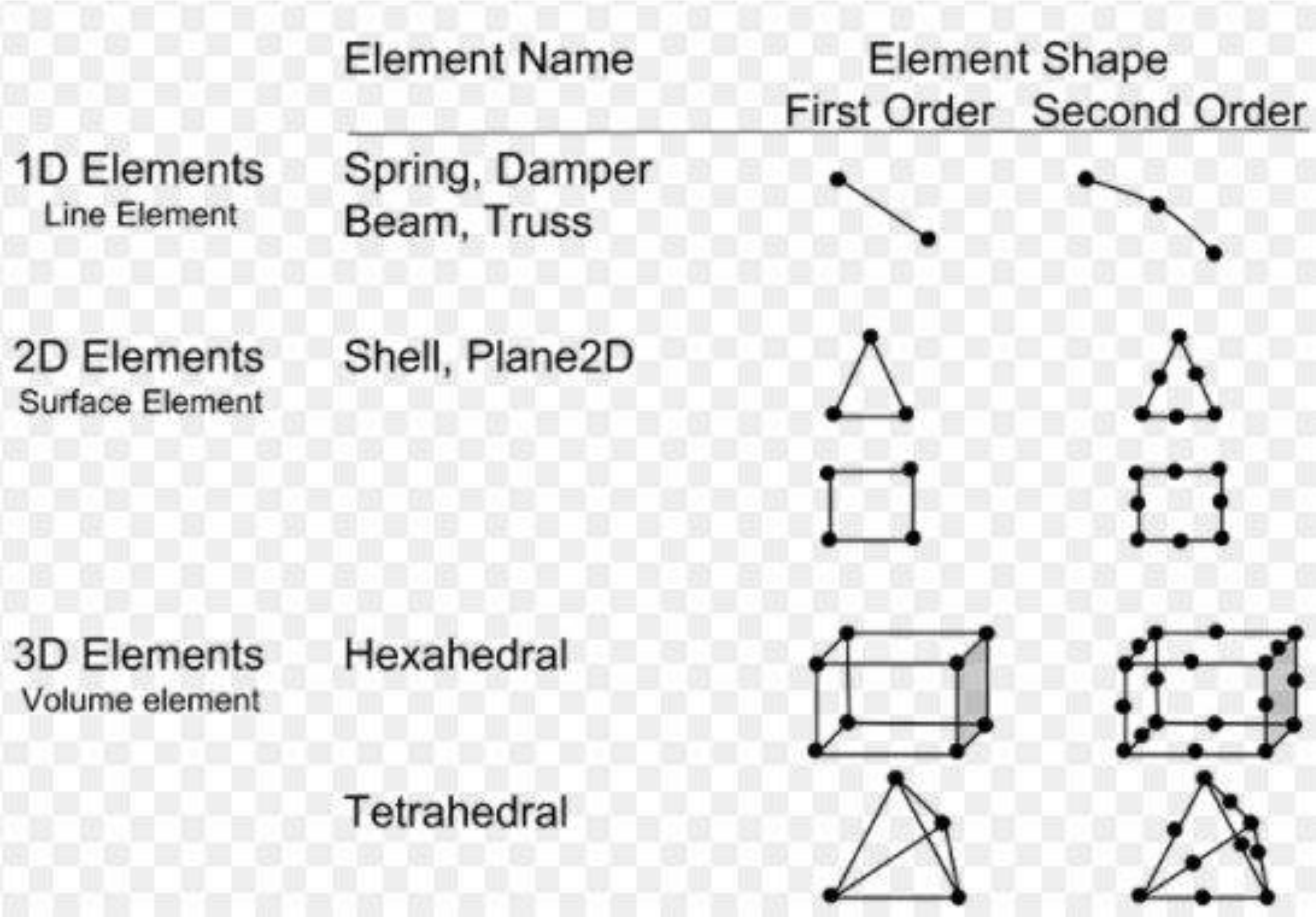
$$v = a_7 + a_8 x + a_9 y + a_{10} x^2 + a_{11} xy + a_{12} y^2$$



(b) Non-linear Quadratic Element

Figure(2): Higher Order Element

If the order of interpolation polynomial of an element is two or more than two such an element is called higher order element. They are complex or multiplex elements, whose order is greater than one. These elements consist both primary as well as secondary nodes. Primary nodes include corner nodes while secondary nodes include internal or mid-point nodes.


Element Name
Element Shape
First Order
Second Order
1D Elements
Line Element
Spring, Damper
Beam, Truss
2D Elements
Surface Element
Shell, Plane2D
3D Elements
Volume element
Hexahedral
Tetrahedral

Isoparametric Element

If the shape or geometry and field displacement variables of the elements are described by the same shape functions of the same order, then the elements are known as isoparametric elements. In isoparametric elements, the number of nodes for defining both geometry and displacements are equal ($i = j$).

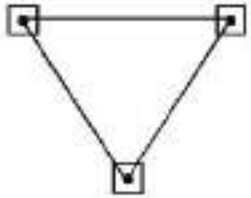


Figure: Isoparametric Element

• → Nodes for defining displacements

□ → Nodes for defining geometry

Two dimensional and three dimensional elasticity problems can be solved using the isoparametric elements.

The displacements can be written as,

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3$$

$$v = N_1 v_1 + N_2 v_2 + N_3 v_3$$

Similarly we can write the geometric coordinates:

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3$$

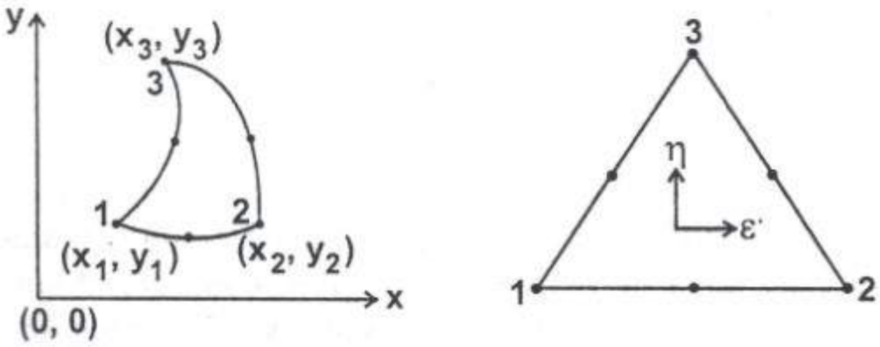
$$y = N_1 y_1 + N_2 y_2 + N_3 y_3$$

Significance of Jacobian Transformation

1. A Jacobian transformation gives the relation between the derivatives in the global and local coordinate systems.
2. It evaluates the deviation of given element from the standard element.
3. It is used to compute the strain displacement matrix.

Most of the problems for which isoparametric elements are employed involves curved boundaries. The finite elements of such problems involves curved sides. These curve sided actual elements are approximated into simple shapes possessing flat surfaces or straight edges. Triangular and quadrilateral elements are the mostly used isoparametric elements.

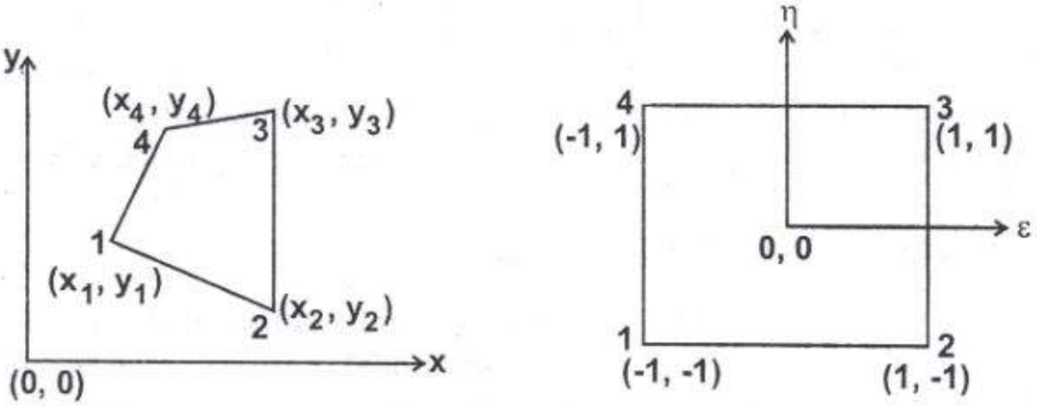
1. Triangular Elements



Figure(1): Two Dimensional Triangular Element

The actual finite triangular element with curved sides, represented in global coordinate system is transformed into straight edged triangular element, represented in natural coordinate system.

2. Quadrilateral Elements

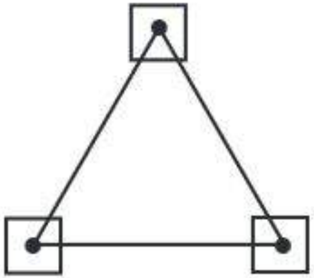
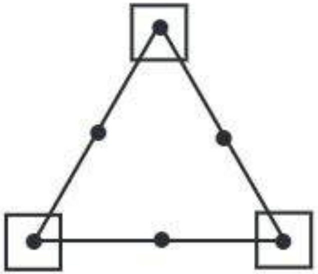
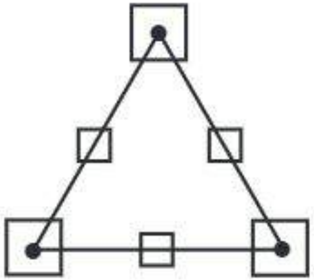


Figure(2): Two Dimensional Quadrilateral Element

The actual finite Quadrilateral element with distorted sides, represented in global coordinate system is transformed into straight edged element, represented in natural coordinate system.

Natural co-ordinates are used to specify a point with in the element with the help of dimensionless numbers whose magnitude do not exceed unity.

Difference between isoparametric, Subparametric, Super parametric elements

Isoparametric Element		Subparametric Element		Superparametric Element	
1.	If the shape or geometry and field displacement variables of the elements are described by same shape functions of the same order, then the elements are known as isoparametric elements.	1.	If the geometry of the elements are described by lower order shape functions compared to field variables (displacements), then the elements are known as subparametric element	1.	If the geometry of the element is described by higher order shape functions compared to field variables (displacements) then the elements are known as super parametric elements.
2.	The number of nodes for defining both geometry and displacements are equal.	2.	The number of nodes for defining the displacements is more than the number of nodes for defining geometry	2.	The number of nodes for defining the displacement is less than the number of nodes for defining geometry
3.	 **Figure: Isoparametric element**	3.	 **Figure: Subparametric element**	3.	 **Figure: Superparametric element**

Consider the general quadrilateral element as shown in figure. The local nodes are numbered as 1, 2, 3 and 4

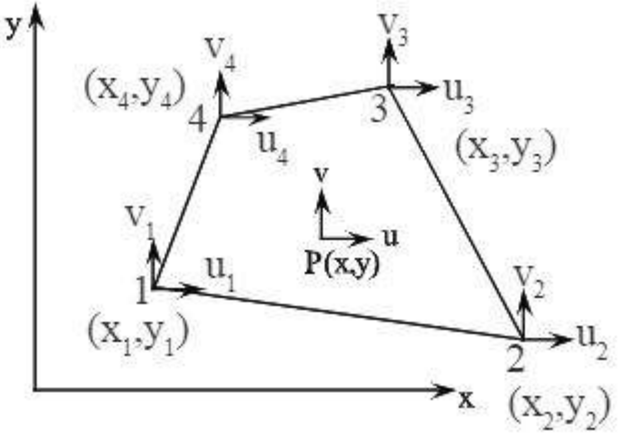


Figure (1) : Four-Noded Element (Global Co-ordinate System)

Four-node quadrilateral element.

Let $(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)$ are the co-ordinates at nodes 1, 2, 3, 4.

u_1, u_2, u_3, u_4 – Displacements along x–axis at nodes 1, 2, 3, 4.

v_1, v_2, v_3, v_4 – Displacements along y–axis at nodes 1, 2, 3, 4., The quadrilateral element in ξ, η space (the master element).

$$u(\xi,\eta) = a_1 + a_2\xi + a_3\eta + a_4\xi\eta$$

$$v(\xi,\eta) = a_5 + a_6\xi + a_7\eta + a_8\xi\eta$$

$a_1, a_2, a_3, \ldots a_8$ – Polynomial coefficients.

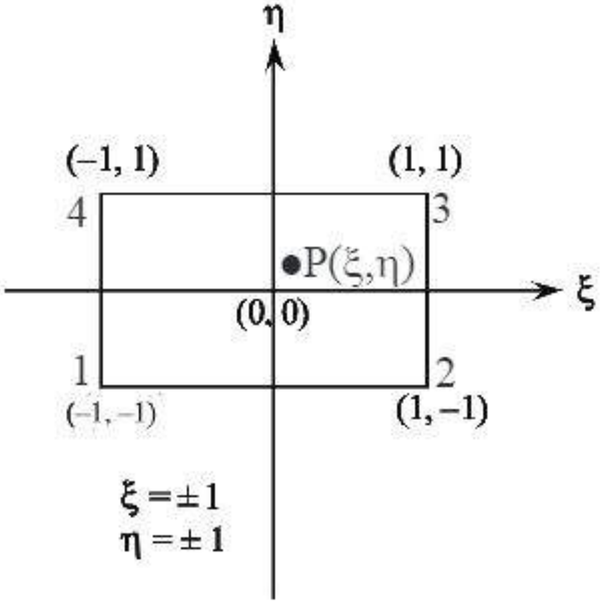
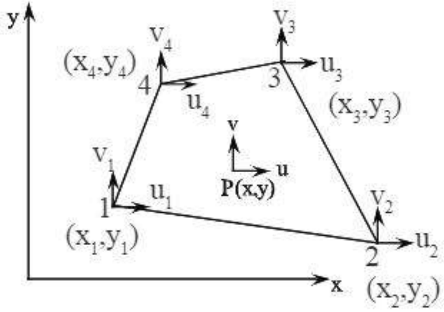
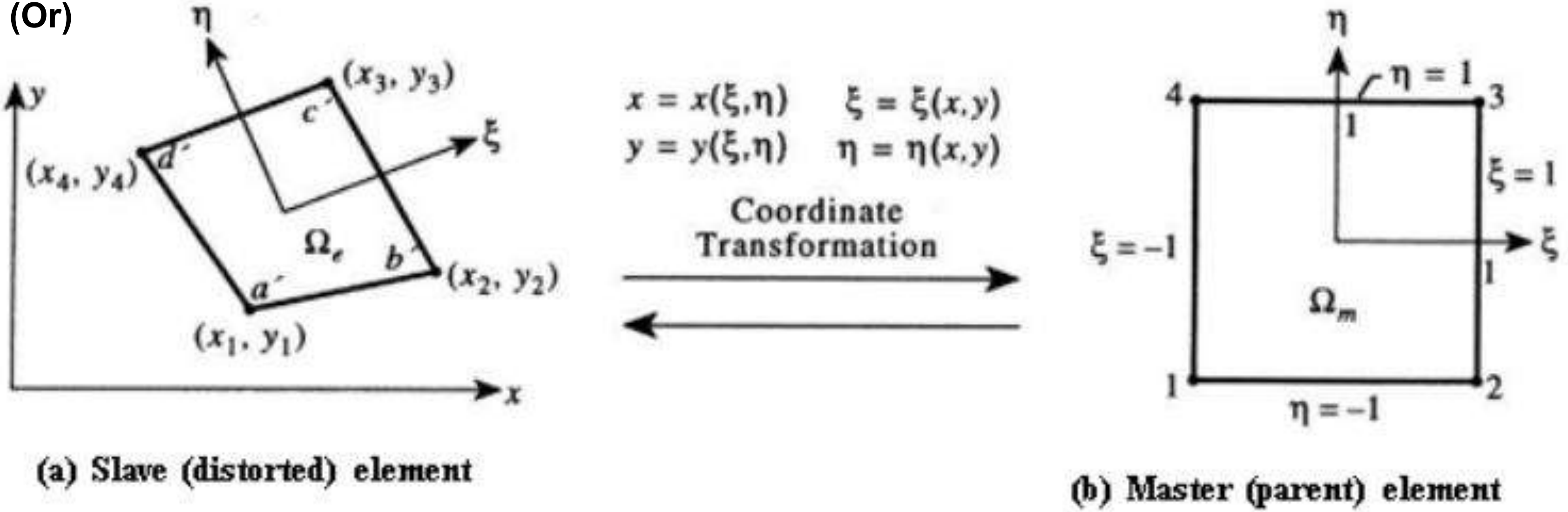


Figure (2): Isoparametric Quadrilateral Element (Natural Co-ordinate System)



(Or)



(a) Slave (distorted) element

(b) Master (parent) element

Isoparametric coordinate transformation.

$$u = a_1 + a_2\xi + a_3\eta + a_4\xi\eta$$

After substituting the coordinates, we will get the following form

$$u = \frac{1}{4}(1-\xi)(1-\eta)u_1 + \frac{1}{4}(1+\xi)(1-\eta)u_2 + \frac{1}{4}(1+\xi)(1+\eta)u_3 + \frac{1}{4}(1-\xi)(1+\eta)u_4$$

The above equation can be written as,

$$\boxed{u = N_1u_1 + N_2u_2 + N_3u_3 + N_4u_4}$$

$$N_i = \begin{cases} 1 \; at \; node \, i \\ 0 \; at \; all \; other \; nodes \end{cases}$$

Where,

N_1, N_2, N_3, N_4 – Shape functions of the isoparametric element

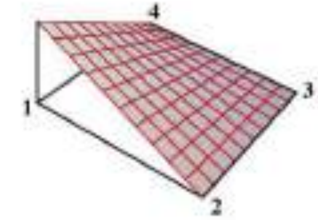
$$N_1 = \frac{1}{4}(1-\xi)(1-\eta)$$

$$N_2 = \frac{1}{4}(1+\xi)(1-\eta)$$

$$N_3 = \frac{1}{4}(1+\xi)(1+\eta)$$

$$N_4 = \frac{1}{4}(1-\xi)(1+\eta)$$

The Four Node Bilinear Quad: Shape Function Plot



$$N_1^e = \tfrac{1}{4}(1-\xi)(1-\eta)$$

$$\begin{aligned} N_1 &= 1 \quad \text{at node 1} \\ &= 0 \quad \text{at nodes 2, 3, and 4} \end{aligned}$$

Similarly $v = a_5 + a_6\,\xi + a_7\eta + a_8\,\xi\eta$ becomes $v = N_1 v_1 + N_2 v_2 + N_3 v_3 + N_4 v_4$

The displacements are

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3 + N_4 u_4$$

$$v = N_1 v_1 + N_2 v_2 + N_3 v_3 + N_4 v_4$$

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

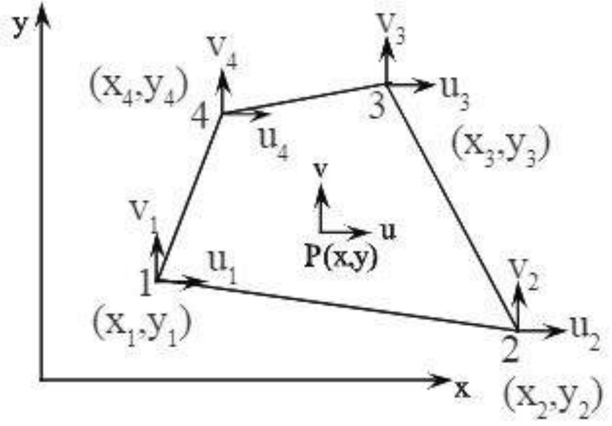
$$= [N]\,\{\delta\}$$

$[N]$ – Shape function matrix

$\{\delta\}$ – Nodal displacement vector

Displacements of any point P,
inside the quadratic element

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$



Using Isoparametric representation,
we can write the geometric coordinates of Point "P"

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \\ x_4 \\ y_4 \end{Bmatrix}$$

Let,

$$f = f(x,y)$$

$$f = f[x(\xi,\eta),\, y(\xi,\eta)]$$

By chain rule of partial differentiation,

$$\frac{\partial f}{\partial \xi} = \frac{\partial f}{\partial x}\times\frac{\partial x}{\partial \xi} + \frac{\partial f}{\partial y}\times\frac{\partial y}{\partial \xi}$$

$$\frac{\partial f}{\partial \eta} = \frac{\partial f}{\partial x}\times\frac{\partial x}{\partial \eta} + \frac{\partial f}{\partial y}\times\frac{\partial y}{\partial \eta}$$

Above equations in matrix form can be expressed as,

$$\begin{Bmatrix} \dfrac{\partial f}{\partial \xi} \\ \dfrac{\partial f}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \end{Bmatrix}$$

This is known as the **Jacobian** matrix (J) for the mapping $(\xi,\eta) \rightarrow (x,y)$

[J]– Jacobian matrix

$$[J] = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

From equations,

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4$$

$$y = N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4$$

$$[J] = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

$$\left\{ \begin{aligned} J_{11} &= \frac{\partial x}{\partial \xi} = \frac{\partial N_1}{\partial \xi} x_1 + \frac{\partial N_2}{\partial \xi} x_2 + \frac{\partial N_3}{\partial \xi} x_3 + \frac{\partial N_4}{\partial \xi} x_4 \\ J_{12} &= \frac{\partial y}{\partial \xi} = \frac{\partial N_1}{\partial \xi} y_1 + \frac{\partial N_2}{\partial \xi} y_2 + \frac{\partial N_3}{\partial \xi} y_3 + \frac{\partial N_4}{\partial \xi} y_4 \\ J_{21} &= \frac{\partial x}{\partial \eta} = \frac{\partial N_1}{\partial \eta} x_1 + \frac{\partial N_2}{\partial \eta} x_2 + \frac{\partial N_3}{\partial \eta} x_3 + \frac{\partial N_4}{\partial \eta} x_4 \\ J_{22} &= \frac{\partial y}{\partial \eta} = \frac{\partial N_1}{\partial \eta} y_1 + \frac{\partial N_2}{\partial \eta} y_2 + \frac{\partial N_3}{\partial \eta} y_3 + \frac{\partial N_4}{\partial \eta} y_4 \end{aligned} \right.$$

Shape functions for an isoparametric Quadrilateral element is given by,

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta)$$

$$N_2 = \frac{1}{4}(1+\xi)(1-\eta)$$

$$N_3 = \frac{1}{4}(1+\xi)(1+\eta)$$

$$N_4 = \frac{1}{4}(1-\xi)(1+\eta)$$

$$J_{11} = \frac{\partial x}{\partial \xi} = \frac{\partial N_1}{\partial \xi}x_1 + \frac{\partial N_2}{\partial \xi}x_2 + \frac{\partial N_3}{\partial \xi}x_3 + \frac{\partial N_4}{\partial \xi}x_4$$

$$J_{12} = \frac{\partial y}{\partial \xi} = \frac{\partial N_1}{\partial \xi}y_1 + \frac{\partial N_2}{\partial \xi}y_2 + \frac{\partial N_3}{\partial \xi}y_3 + \frac{\partial N_4}{\partial \xi}y_4$$

$$J_{21} = \frac{\partial x}{\partial \eta} = \frac{\partial N_1}{\partial \eta}x_1 + \frac{\partial N_2}{\partial \eta}x_2 + \frac{\partial N_3}{\partial \eta}x_3 + \frac{\partial N_4}{\partial \eta}x_4$$

$$J_{22} = \frac{\partial y}{\partial \eta} = \frac{\partial N_1}{\partial \eta}y_1 + \frac{\partial N_2}{\partial \eta}y_2 + \frac{\partial N_3}{\partial \eta}y_3 + \frac{\partial N_4}{\partial \eta}y_4$$

$$\frac{\partial N_1}{\partial \xi} = \frac{1}{4}(-1)(1-\eta) = \frac{-1}{4}(1-\eta)$$

$$\frac{\partial N_2}{\partial \xi} = \frac{1}{4}(1)(1-\eta) = \frac{1}{4}(1-\eta)$$

$$\frac{\partial N_3}{\partial \xi} = \frac{1}{4}(1)(1+\eta) = \frac{1}{4}(1+\eta)$$

$$\frac{\partial N_4}{\partial \xi} = \frac{1}{4}(-1)(1+\eta) = \frac{-1}{4}(1+\eta)$$

$$\frac{\partial N_1}{\partial \eta} = \frac{1}{4}(1-\xi)(-1) = \frac{-1}{4}(1-\xi)$$

$$\frac{\partial N_2}{\partial \eta} = \frac{1}{4}(1+\xi)(-1) = \frac{-1}{4}(1+\xi)$$

$$\frac{\partial N_3}{\partial \eta} = \frac{1}{4}(1+\xi)(1) = \frac{1}{4}(1+\xi)$$

$$\frac{\partial N_4}{\partial \eta} = \frac{1}{4}(1-\xi)(1) = \frac{1}{4}(1-\xi)$$

$$\begin{Bmatrix} \dfrac{\partial f}{\partial \xi} \\ \dfrac{\partial f}{\partial \eta} \end{Bmatrix} = [\mathrm{J}] \begin{Bmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \end{Bmatrix}$$

$$\begin{Bmatrix} \dfrac{\partial f}{\partial \xi} \\ \dfrac{\partial f}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \end{Bmatrix}$$

$$\begin{Bmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \end{Bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}^{-1} \begin{Bmatrix} \dfrac{\partial f}{\partial \xi} \\ \dfrac{\partial f}{\partial \eta} \end{Bmatrix}$$

$$\begin{Bmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \end{Bmatrix} = \frac{1}{|J|} \begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial f}{\partial \xi} \\ \dfrac{\partial f}{\partial \eta} \end{Bmatrix}$$

$$\begin{Bmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \end{Bmatrix} = \frac{1}{|J|} \begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial f}{\partial \xi} \\ \dfrac{\partial f}{\partial \eta} \end{Bmatrix}$$

Replacing f with u and v separately we get,

$$\begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{Bmatrix} = \frac{1}{|J|} \begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \end{Bmatrix}$$

$$\begin{Bmatrix} \dfrac{\partial v}{\partial x} \\ \dfrac{\partial v}{\partial y} \end{Bmatrix} = \frac{1}{|J|} \begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial v}{\partial \xi} \\ \dfrac{\partial v}{\partial \eta} \end{Bmatrix}$$

Using the strain–displacement relations

$$\boldsymbol{\epsilon} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix} = \frac{1}{|J|}\begin{bmatrix} J_{22} & -J_{12} & 0 & 0 \\ 0 & 0 & -J_{21} & J_{11} \\ -J_{22} & J_{11} & J_{22} & -J_{12} \end{bmatrix}\begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \\ \dfrac{\partial v}{\partial \xi} \\ \dfrac{\partial v}{\partial \eta} \end{Bmatrix}$$

$$\boldsymbol{\epsilon} = \begin{Bmatrix} \boldsymbol{\epsilon}_x \\ \boldsymbol{\epsilon}_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix}$$

From equations,

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3 + N_4 u_4$$

$$v = N_1 v_1 + N_2 v_2 + N_3 v_3 + N_4 v_4$$

Differentiating the above equations w.r. to 'ξ' and 'η',

$$\frac{\partial u}{\partial \xi} = \frac{\partial N_1}{\partial \xi} u_1 + \frac{\partial N_2}{\partial \xi} u_2 + \frac{\partial N_3}{\partial \xi} u_3 + \frac{\partial N_4}{\partial \xi} u_4$$

$$\frac{\partial u}{\partial \eta} = \frac{\partial N_1}{\partial \eta} u_1 + \frac{\partial N_2}{\partial \eta} u_2 + \frac{\partial N_3}{\partial \eta} u_3 + \frac{\partial N_4}{\partial \eta} u_4$$

$$\frac{\partial v}{\partial \xi} = \frac{\partial N_1}{\partial \xi} v_1 + \frac{\partial N_2}{\partial \xi} v_2 + \frac{\partial N_3}{\partial \xi} v_3 + \frac{\partial N_4}{\partial \xi} v_4$$

$$\frac{\partial v}{\partial \eta} = \frac{\partial N_1}{\partial \eta} v_1 + \frac{\partial N_2}{\partial \eta} v_2 + \frac{\partial N_3}{\partial \eta} v_3 + \frac{\partial N_4}{\partial \eta} v_4$$

$$\boxed{\begin{Bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \\ \frac{\partial v}{\partial \xi} \\ \frac{\partial v}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \frac{\partial N_1}{\partial \xi} & 0 & \frac{\partial N_2}{\partial \xi} & 0 & \frac{\partial N_3}{\partial \xi} & 0 & \frac{\partial N_4}{\partial \xi} & 0 \\ \frac{\partial N_1}{\partial \eta} & 0 & \frac{\partial N_2}{\partial \eta} & 0 & \frac{\partial N_3}{\partial \eta} & 0 & \frac{\partial N_4}{\partial \eta} & 0 \\ 0 & \frac{\partial N_1}{\partial \xi} & 0 & \frac{\partial N_2}{\partial \xi} & 0 & \frac{\partial N_3}{\partial \xi} & 0 & \frac{\partial N_4}{\partial \xi} \\ 0 & \frac{\partial N_1}{\partial \eta} & 0 & \frac{\partial N_2}{\partial \eta} & 0 & \frac{\partial N_3}{\partial \eta} & 0 & \frac{\partial N_4}{\partial \eta} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}}$$

$$\boldsymbol{\epsilon} = \begin{Bmatrix} \boldsymbol{\epsilon}_x \\ \boldsymbol{\epsilon}_y \\ \gamma_{xy} \end{Bmatrix} = \underbrace{\frac{1}{|J|}\begin{bmatrix} J_{22} & -J_{12} & 0 & 0 \\ 0 & 0 & -J_{21} & J_{11} \\ -J_{21} & J_{11} & J_{22} & -J_{12} \end{bmatrix} \times \begin{bmatrix} \frac{\partial N_1}{\partial \xi} & 0 & \frac{\partial N_2}{\partial \xi} & 0 & \frac{\partial N_3}{\partial \xi} & 0 & \frac{\partial N_4}{\partial \xi} & 0 \\ \frac{\partial N_1}{\partial \eta} & 0 & \frac{\partial N_2}{\partial \eta} & 0 & \frac{\partial N_3}{\partial \eta} & 0 & \frac{\partial N_4}{\partial \eta} & 0 \\ 0 & \frac{\partial N_1}{\partial \xi} & 0 & \frac{\partial N_2}{\partial \xi} & 0 & \frac{\partial N_3}{\partial \xi} & 0 & \frac{\partial N_4}{\partial \xi} \\ 0 & \frac{\partial N_1}{\partial \eta} & 0 & \frac{\partial N_2}{\partial \eta} & 0 & \frac{\partial N_3}{\partial \eta} & 0 & \frac{\partial N_4}{\partial \eta} \end{bmatrix}}_{\textbf{[B]}} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

Strain Displacement Matrix

$$[B] = \frac{1}{|J|}\begin{bmatrix} J_{22} & -J_{12} & 0 & 0 \\ 0 & 0 & -J_{21} & J_{11} \\ -J_{21} & J_{11} & J_{22} & -J_{12} \end{bmatrix} \times \frac{1}{4}\begin{bmatrix} -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) & 0 \\ -(1-\xi) & 0 & -(1+\xi) & 0 & (1+\xi) & 0 & (1-\xi) & 0 \\ 0 & -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) \\ 0 & -(1-\xi) & 0 & -(1+\xi) & 0 & (1+\xi) & 0 & (1-\xi) \end{bmatrix}$$

Element stiffness matrix is given by,

$$[K] = \oint [B]^T [D] [B] dv$$

$$[k] = \iint_A [B]^T [D][B]\, t\, dx\, dy$$

Where,

$[B]$– Strain-displacement matrix

$[D]$– Stress-strain relationship matrix

t– Thickness of the element

$$[k] = \iint_A [B]^T [D][B]\, t\, dx\, dy$$

The above equation is in global co-ordinate system.

$$\boxed{\iint_A f(x,y)\,dx\,dy = \iint_A f(\xi,\eta)\,|J|\,d\xi\,d\eta}$$

In natural co-ordinate system, stiffness matrix is given by,

$$[k] = \int_{-1}^{1}\int_{-1}^{1} [B]^T [D][B]\, t\, |J|\, d\xi\, d\eta$$

Element Stress {σ}

$\{\sigma\} = [D][B]\{\delta\}$

$[D]$ – Stress-strain relationship matrix

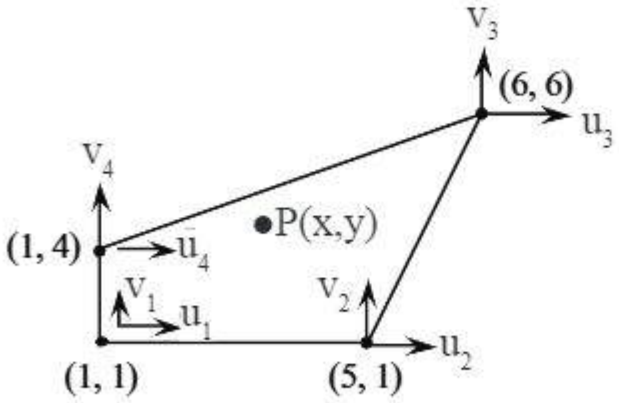
Q. Figure shows a four-node quadrilateral. The (x, y) co-ordinates of each node are given in the figure. The element displacement vector $\{\delta\}$ is given as,

$$\{\delta\} = [\, 0, 0, 0.20, 0, 0.15, 0.10, 0, 0.05]^T$$

Find the following,

(a) The x, y-coordinates of a point P whose location in the master element is given by $\xi = 0.5$ and $\eta = 0.5$ and

(b) The u, v displacements of the point P.



Figure

Given that,

Displacement vector,

$$\{\delta\} = \begin{array}{cccccccc} u_1 & v_1 & u_2 & v_2 & u_3 & v_3 & u_4 & v_4 \\ [0, & 0, & 0.20, & 0, & 0.15 & 0.10, & 0, & 0.05]^T \end{array}$$

$\eta = \xi = 0.5 \Rightarrow \mathbf{P}(\xi, \eta)$

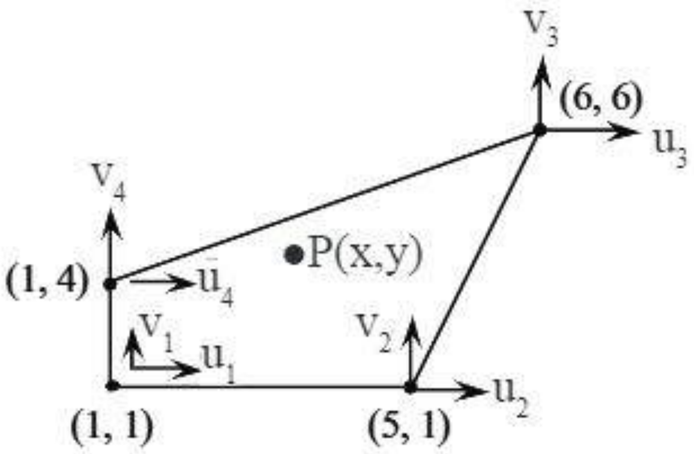
The coordinates of the four noded quadratic element are,

$x_1 = 1 \quad ; \quad y_1 = 1$

$x_2 = 5 \quad ; \quad y_2 = 1$

$x_3 = 6 \quad ; \quad y_3 = 6$

$x_4 = 1 \quad ; \quad y_4 = 4$



Figure

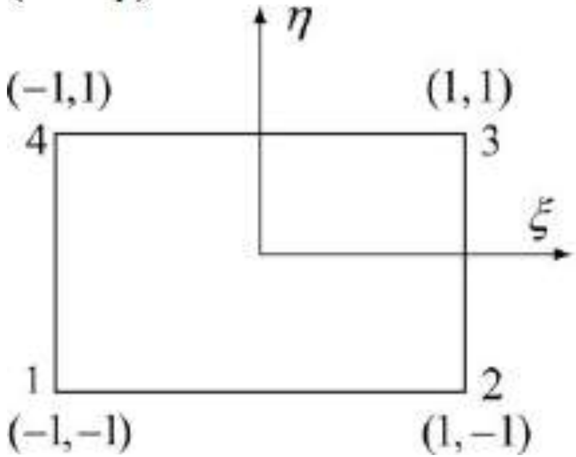
(a) Coordinates of Point 'P'

The coordinates of point $P(x, y)$ are given by,

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4$$

$$y = N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4$$

$$N_4 = \frac{1}{4}(1-\xi)(1+\eta) \qquad N_3 = \frac{1}{4}(1+\xi)(1+\eta)$$



$$N_1 = \frac{1}{4}(1-\xi)(1-\eta) \qquad N_2 = \frac{1}{4}(1+\xi)(1-\eta)$$

Shape functions of a four noded quadratic element are,

$$N_1 = \frac{(1-\xi)(1-\eta)}{4} = \frac{(1-0.5)(1-0.5)}{4} = 0.0625$$

$$N_2 = \frac{(1+\xi)(1-\eta)}{4} = \frac{(1+0.5)(1-0.5)}{4} = 0.1875$$

$$N_3 = \frac{(1+\xi)(1+\eta)}{4} = \frac{(1+0.5)(1+0.5)}{4} = 0.5625$$

$$N_4 = \frac{(1-\xi)(1+\eta)}{4} = \frac{(1-0.5)(1+0.5)}{4} = 0.1875$$

$$\therefore \quad x = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4$$

$$= 0.0625\,(1) + 0.1875\,(5) + 0.5625\,(6) + 0.1875\,(1)$$

$$x = 4.5625$$

$$y = N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4$$

$$= 0.0625(1) + 0.1875\,(1) + 0.5625\,(6) + 0.1875\,(4)$$

$$= 4.375$$

$$\boxed{\therefore \; P(x, y) = (4.5625,\ 4.375)}$$

(b) Displacements of Point 'P'

u, v displacements of point 'P' are given by,

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3 + N_4 u_4$$

$$= 0.0625(0) + 0.1875(0.20) + 0.5625\,(0.15) + 0.1875\,(0)$$

$$= 0.121875$$

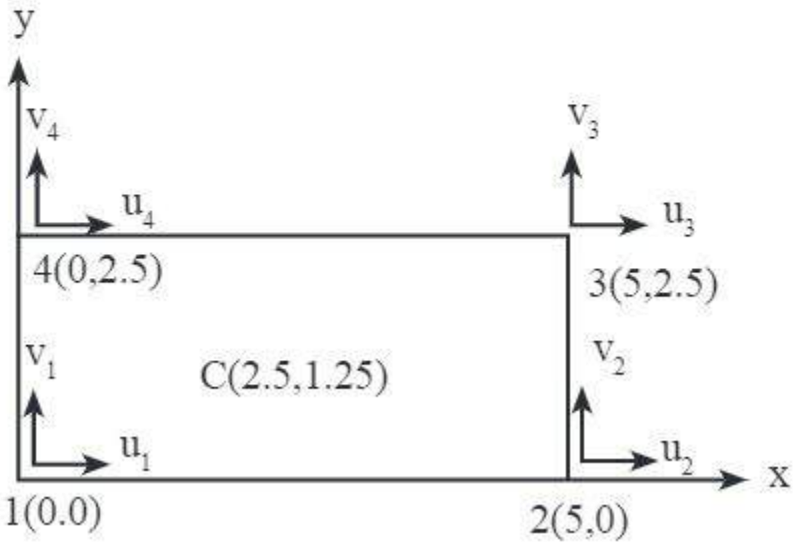
$$v = N_1 v_1 + N_2 v_2 + N_3 v_3 + N_4 v_4$$

$$= 0.0625\,(0) + 0.1875\,(0) + 0.5625\,(0.10) + 0.1875\,(0.05)$$

$$v = 0.065625$$

$$\boxed{\therefore\ (u, v) = (0.121875,\ 0.065625)}$$

Q. Consider a rectangular element as shown in figure. Assume plane stress condition, E = 206850 MPa, $\nu = 0.3$, $\{\delta\} = [0, 0, 0.05, 0.075, 0.15, 0.8, 0, 0]^T$ cm. Evaluate Jacobian J, B and σ at $\xi = 0$ and $\eta = 0$.



Given that,

Young's modulus, $E = 206850$ MPa

$$= 206850 \text{ N/mm}^2$$

$$E = 20.685 \times 10^6 \text{ N/cm}^2$$

Poisons ratio, $\upsilon = 0.3$

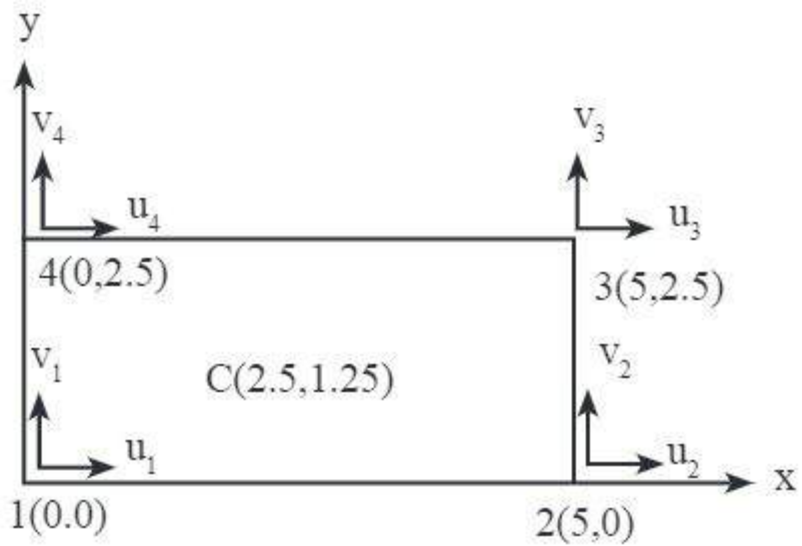
Displacement vector,

$$\begin{array}{cccccccccc} & u_1 & v_1 & u_2 & v_2 & u_3 & v_3 & u_4 & v_4 & \\ \{\delta\} = [& 0, & 0, & 0.05, & 0.075, & 0.15 & 0.8, & 0, & 0]^T & \text{cm} \end{array}$$

Local coordinates, $\xi = 0, \eta = 0$

For a plane stress condition,

$$[D] = \frac{E}{1-v^2}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix}$$



Co-ordinates of the Quadratic element is

$$x_1 = 0, y_1 = 0$$

$$x_2 = 5, y_2 = 0$$

$$x_3 = 5, y_3 = 2.5$$

$$x_4 = 0, y_4 = 2.5$$

Jacobian J, B and σ

[J]– Jacobian matrix

$$[J]=\begin{bmatrix}\frac{\partial x}{\partial\xi} & \frac{\partial y}{\partial\xi}\\ \frac{\partial x}{\partial\eta} & \frac{\partial y}{\partial\eta}\end{bmatrix}=\begin{bmatrix}J_{11} & J_{12}\\ J_{21} & J_{22}\end{bmatrix}$$

Where,

$$x=N_1x_1+N_2x_2+N_3x_3+N_4x_4$$
$$y=N_1y_1+N_2y_2+N_3y_3+N_4y_4$$

$$N_1=\frac{1}{4}(1-\xi)(1-\eta)$$
$$N_2=\frac{1}{4}(1+\xi)(1-\eta)$$
$$N_3=\frac{1}{4}(1+\xi)(1+\eta)$$
$$N_4=\frac{1}{4}(1-\xi)(1+\eta)$$

$$\frac{\partial N_1}{\partial\xi}=\frac{1}{4}(-1)(1-\eta)=\frac{-1}{4}(1-\eta)$$
$$\frac{\partial N_2}{\partial\xi}=\frac{1}{4}(1)(1-\eta)=\frac{1}{4}(1-\eta)$$
$$\frac{\partial N_3}{\partial\xi}=\frac{1}{4}(1)(1+\eta)=\frac{1}{4}(1+\eta)$$
$$\frac{\partial N_4}{\partial\xi}=\frac{1}{4}(-1)(1+\eta)=\frac{-1}{4}(1+\eta)$$
$$\frac{\partial N_1}{\partial\eta}=\frac{1}{4}(1-\xi)(-1)=\frac{-1}{4}(1-\xi)$$
$$\frac{\partial N_2}{\partial\eta}=\frac{1}{4}(1+\xi)(-1)=\frac{-1}{4}(1+\xi)$$
$$\frac{\partial N_3}{\partial\eta}=\frac{1}{4}(1+\xi)(1)=\frac{1}{4}(1+\xi)$$
$$\frac{\partial N_4}{\partial\eta}=\frac{1}{4}(1-\xi)(1)=\frac{1}{4}(1-\xi)$$

$$J_{11} = \frac{1}{4}[-(1-\eta)x_1 + (1-\eta)x_2 + (1+\eta)x_3 - (1+\eta)x_4]$$

$$= \frac{1}{4}[(1-0)0 + (1-0)5 + (1+0)5 - (1+0)0]$$

$$= \frac{1}{4}(5+5)$$

$$J_{11} = 2.5$$

$$J_{12} = \frac{1}{4}[-(1-\eta)y_1 + (1-\eta)y_2 + (1+\eta)y_3 - (1+\eta)y_4]$$

$$= \frac{1}{4}[-(1-0)0 + (1-0)0 + (1+0)2.5 - (1+0)2.5]$$

$$J_{12} = 0$$

$$J_{21} = \frac{1}{4}[-(1-\xi)x_1 - (1+\xi)x_2 + (1+\xi)x_3 + (1-\xi)x_4]$$

$$\frac{1}{4}[-(1-0)0 - (1+0)5 + (1+0)5 + (1-0)0]$$

$$J_{21} = 0$$

$$J_{22} = \frac{1}{4}[-(1-\xi)y_1 - (1+\xi)y_2 + (1+\xi)y_3 + (1-\xi)y_4]$$

$$= \frac{1}{4}[-(1-0)0 - (1+0)0 + (1+0)2.5 + (1-0)2.5]$$

$$= \frac{1}{4}(5)$$

$$J_{22} = 1.25$$

Jacobian Matrix,

$$[J] = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} = \begin{bmatrix} 2.5 & 0 \\ 0 & 1.25 \end{bmatrix}$$

$$\begin{Bmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \end{Bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}^{-1} \begin{Bmatrix} \dfrac{\partial f}{\partial \xi} \\ \dfrac{\partial f}{\partial \eta} \end{Bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \underbrace{\frac{1}{ad-bc}}_{\text{determinant}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\begin{Bmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \end{Bmatrix} = \frac{1}{|J|} \begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial f}{\partial \xi} \\ \dfrac{\partial f}{\partial \eta} \end{Bmatrix}$$

Replacing f with u and v separately we get,

$$\begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{Bmatrix} = \frac{1}{|J|} \begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \end{Bmatrix}$$

$$\begin{Bmatrix} \dfrac{\partial v}{\partial x} \\ \dfrac{\partial v}{\partial y} \end{Bmatrix} = \frac{1}{|J|} \begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial v}{\partial \xi} \\ \dfrac{\partial v}{\partial \eta} \end{Bmatrix}$$

$$\boxed{\boldsymbol{\epsilon} = \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix}}$$

$$= \frac{1}{|J|} \begin{bmatrix} J_{22} & -J_{12} & 0 & 0 \\ 0 & 0 & -J_{21} & J_{11} \\ -J_{22} & J_{11} & J_{22} & -J_{12} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \\ \dfrac{\partial v}{\partial \xi} \\ \dfrac{\partial v}{\partial \eta} \end{Bmatrix}$$

Using the strain–displacement relations

$$\boldsymbol{\epsilon} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix} = \frac{1}{|J|}\begin{bmatrix} J_{22} & -J_{12} & 0 & 0 \\ 0 & 0 & -J_{21} & J_{11} \\ -J_{22} & J_{11} & J_{22} & -J_{12} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \\ \dfrac{\partial v}{\partial \xi} \\ \dfrac{\partial v}{\partial \eta} \end{Bmatrix}$$

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3 + N_4 u_4$$

$$v = N_1 v_1 + N_2 v_2 + N_3 v_3 + N_4 v_4$$

$$\frac{\partial u}{\partial \xi} = \frac{\partial N_1}{\partial \xi} u_1 + \frac{\partial N_2}{\partial \xi} u_2 + \frac{\partial N_3}{\partial \xi} u_3 + \frac{\partial N_4}{\partial \xi} u_4$$

$$\begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \\ \dfrac{\partial v}{\partial \xi} \\ \dfrac{\partial v}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial N_1}{\partial \xi} & 0 & \dfrac{\partial N_2}{\partial \xi} & 0 & \dfrac{\partial N_3}{\partial \xi} & 0 & \dfrac{\partial N_4}{\partial \xi} & 0 \\ \dfrac{\partial N_1}{\partial \eta} & 0 & \dfrac{\partial N_2}{\partial \eta} & 0 & \dfrac{\partial N_3}{\partial \eta} & 0 & \dfrac{\partial N_4}{\partial \eta} & 0 \\ 0 & \dfrac{\partial N_1}{\partial \xi} & 0 & \dfrac{\partial N_2}{\partial \xi} & 0 & \dfrac{\partial N_3}{\partial \xi} & 0 & \dfrac{\partial N_4}{\partial \xi} \\ 0 & \dfrac{\partial N_1}{\partial \eta} & 0 & \dfrac{\partial N_2}{\partial \eta} & 0 & \dfrac{\partial N_3}{\partial \eta} & 0 & \dfrac{\partial N_4}{\partial \eta} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

$$\frac{\partial N_1}{\partial \xi} = \frac{1}{4}(-1)(1-\eta) = \frac{-1}{4}(1-\eta)$$

$$\frac{\partial N_2}{\partial \xi} = \frac{1}{4}(1)(1-\eta) = \frac{1}{4}(1-\eta)$$

$$\frac{\partial N_3}{\partial \xi} = \frac{1}{4}(1)(1+\eta) = \frac{1}{4}(1+\eta)$$

$$\frac{\partial N_4}{\partial \xi} = \frac{1}{4}(-1)(1+\eta) = \frac{-1}{4}(1+\eta)$$

$$\frac{\partial N_1}{\partial \eta} = \frac{1}{4}(1-\xi)(-1) = \frac{-1}{4}(1-\xi)$$

$$\frac{\partial N_2}{\partial \eta} = \frac{1}{4}(1+\xi)(-1) = \frac{-1}{4}(1+\xi)$$

$$\frac{\partial N_3}{\partial \eta} = \frac{1}{4}(1+\xi)(1) = \frac{1}{4}(1+\xi)$$

$$\frac{\partial N_4}{\partial \eta} = \frac{1}{4}(1-\xi)(1) = \frac{1}{4}(1-\xi)$$

Strain-Displacement Matrix [B]

$$[B] = \frac{1}{|J|}\begin{bmatrix} J_{22} & -J_{12} & 0 & 0 \\ 0 & 0 & -J_{21} & J_{11} \\ -J_{21} & J_{11} & J_{22} & -J_{12} \end{bmatrix} \times \frac{1}{4}\begin{bmatrix} -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) & 0 \\ -(1-\xi) & 0 & -(1+\xi) & 0 & (1+\xi) & 0 & (1-\xi) & 0 \\ 0 & -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) \\ 0 & -(1-\xi) & 0 & -(1+\xi) & 0 & (1+\xi) & 0 & (1-\xi) \end{bmatrix}$$

$$|J| = [(J_{11} \times J_{22}) - (J_{12} \times J_{21})]$$

$$= (2.5 \times 1.25) - 0$$

$$|J| = 3.125$$

$$[B] = \frac{1}{3.125}\begin{bmatrix} 1.25 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2.5 \\ 0 & 2.5 & 1.25 & 0 \end{bmatrix} \times \frac{1}{4}\begin{bmatrix} -(1-0) & 0 & (1-0) & 0 & (1+0) & 0 & -(1+0) & 0 \\ -(1-0) & 0 & -(1+0) & 0 & (1+0) & 0 & (1-0) & 0 \\ 0 & -(1-0) & 0 & (1-0) & 0 & (1+0) & 0 & -(1+0) \\ 0 & -(1-0) & 0 & -(1+0) & 0 & (1+0) & 0 & (1-0) \end{bmatrix}$$

$$= \frac{1}{3.125}\begin{bmatrix} 1.25 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2.5 \\ 0 & 2.5 & 1.25 & 0 \end{bmatrix} \times \frac{1}{4}\begin{bmatrix} -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ -1 & 0 & -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & -1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

$$
= \begin{bmatrix} 0.4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.8 \\ 0 & 0.8 & 0.4 & 0 \end{bmatrix} \times \begin{bmatrix} -0.25 & 0 & 0.25 & 0 & 0.25 & 0 & -0.25 & 0 \\ -0.25 & 0 & -0.25 & 0 & 0.25 & 0 & 0.25 & 0 \\ 0 & -0.25 & 0 & 0.25 & 0 & 0.25 & 0 & -0.25 \\ 0 & -0.25 & 0 & -0.25 & 0 & 0.25 & 0 & 0.25 \end{bmatrix}
$$

$$
= \begin{bmatrix} (0.4\times-0.25) & 0 & (0.4\times0.25) & 0 & (0.4\times2.5) & 0 & (0.4\times-0.25) & 0 \\ 0 & (0.8\times-0.25) & 0 & (0.8\times-0.25) & 0 & (0.8\times0.25) & 0 & (0.8\times0.25) \\ (0.8\times-0.25) & (0.4\times-0.25) & (0.8\times0.25) & (0.4\times0.25) & (0.8\times0.25) & (0.4\times0.25) & (0.8\times0.25) & (0.4\times-0.25) \end{bmatrix}
$$

$$
\boxed{[B] = \begin{bmatrix} -0.1 & 0 & 0.1 & 0 & 0.1 & 0 & -0.1 & 0 \\ 0 & -0.2 & 0 & -0.2 & 0 & 0.2 & 0 & 0.2 \\ -0.2 & -0.1 & -0.2 & 0.1 & 0.2 & 0.1 & 0.2 & -0.1 \end{bmatrix}}
$$

Element Stress {σ}

$$\{\sigma\} = [D][B]\{\delta\}$$

$[D]$ – Stress-strain relationship matrix

For a plane stress condition,

$$[D] = \frac{E}{1-v^2}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix} = \frac{20.685\times 10^6}{1-(0.3)^2}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & \frac{1-0.3}{2} \end{bmatrix}$$

$$\boxed{[D] = 22.73 \times 10^6 \begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}}$$

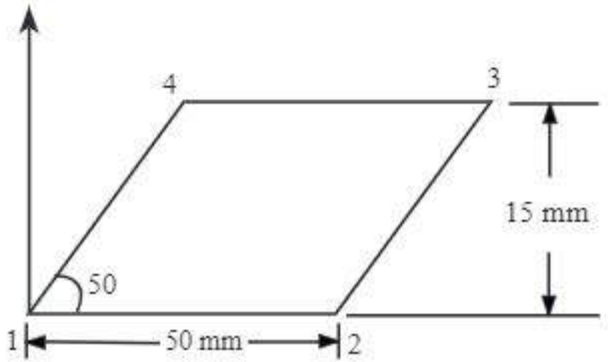
$$\{\sigma\} = 22.731 \times 10^6 \begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}_{3\times 3} \begin{bmatrix} -0.1 & 0 & 0.1 & 0 & 0.1 & 0 & -0.1 & 0 \\ 0 & -0.2 & 0 & -0.2 & 0 & 0.2 & 0 & 0.2 \\ -0.2 & -0.1 & -0.2 & 0.1 & 0.2 & 0.1 & 0.2 & -0.1 \end{bmatrix}_{3\times 8} \begin{Bmatrix} 0 \\ 0 \\ 0.05 \\ 0.075 \\ 0.15 \\ 0.8 \\ 0 \\ 0 \end{Bmatrix}_{8\times 1}$$

$$= 22.731\times 10^6 \begin{bmatrix} -0.1 & -0.06 & 0.1 & -0.06 & 0.1 & 0.06 & -0.1 & 0.06 \\ -0.03 & -0.2 & 0.03 & -0.2 & 0.03 & 0.2 & -0.03 & 0.2 \\ -0.07 & -0.035 & -0.07 & 0.035 & 0.07 & 0.035 & 0.07 & -0.035 \end{bmatrix}_{3\times 8} \begin{Bmatrix} 0 \\ 0 \\ 0.05 \\ 0.075 \\ 0.015 \\ 0.8 \\ 0 \\ 0 \end{Bmatrix}_{8\times 1}$$

$$= 22.731\times 10^6 \begin{bmatrix} 0.05 \\ 0.14695 \\ 0.028175 \end{bmatrix}$$

$$\{\sigma\} = \begin{Bmatrix} 1.136\times 10^6 \\ 3.34\times 10^6 \\ 0.64\times 10^6 \end{Bmatrix} \text{N/cm}^2$$

Consider a quadrilateral element as shown in figure. The local coordinates are ξ = 0.5, η = 0.5, Evaluate Jacobian matrix and strain - Displacement matrix.



Figure

Numerical Integration

Element stiffness matrix is given by,

$$[K] = \oint [B]^T [D][B] \, dv$$

$$[k] = \iint_A [B]^T [D][B] \, t \, dx \, dy$$

$$\boxed{\iint_A f(x,y) \, dx \, dy = \iint_A f(\xi,\eta) \, |J| \, d\xi \, d\eta}$$

$$\boxed{[k] = \int_{-1}^{1} \int_{-1}^{1} [B]^T [D][B] \, t \, |J| \, d\xi \, d\eta}$$

Consider Gauss Quadrature formula,

$$I = \int_a^b f(x)\,dx = \sum_{i=1}^{n} W_i f(x_i)$$

Where,

W_i – Weight function

x_i – Sampling point

one point technique

Let, $n = 1$.

$$I = \int_{-1}^{1} f(x)\,dx = W_1 f(x_1)$$

The polynomial order is $(2n - 1)$

$$= (2(1) - 1)$$

$$= 1$$

Therefore, the polynomial function is given by,

$$f(x) = a_1 + a_2 x$$

Gauss quadrature of linear polynomial

$$I = \int_{-1}^{1} (a_1 + a_2 x)dx = W_1 f(x_1)$$

$$\int_{-1}^{1} (a_1 + a_2 x)dx - W_1 f(x_1) = 0$$

$$\int_{-1}^{1} (a_1 + a_2 x)dx - W_1 (a_1 + a_2 x_1) = 0$$

$$a_1 [x]_{-1}^{1} + a_2 \left[\frac{x^2}{2}\right]_{-1}^{1} - W_1 (a_1 + a_2 x_1) = 0$$

$$2a_1 - W_1 a_1 - W_1 a_2 x_1 = 0$$

$$a_1 (2 - W_1) - W_1 a_2 x_1 = 0$$

Let, $n = 1$.

$$I = \int_{-1}^{1} f(x)dx = W_1 f(x_1)$$

satisfied only if $W_1 = 2$ or $x = 0$.

$$I = \int_{-1}^{1} f(x)dx = 2f(0)$$

Two Point Technique (Formula)

Consider Gauss Quadrature formula,

$$I = \int_a^b f(x)\,dx = \sum_{i=1}^{n} W_i f(x_i)$$

Let, $n = 2$.

$$I = \int_{-1}^{1} f(x)\,dx = W_1 f(x_1) + W_2 f(x_2)$$

The polynomial order is $(2n - 1)$

$$= (2(2) - 1)$$

$$= 3$$

Gauss quadrature of Cubic polynomial

Therefore, the polynomial function is given by,

$$f(x) = a_1 + a_2 x + a_3 x^2 + a_4 x^3$$

$$\int_{-1}^{1} \left(a_1 + a_2 x + a_3 x^2 + a_4 x^3\right) dx = W_1 f(x_1) + W_2 f(x_2)$$

$$\left[a_1 x + a_2 \frac{x^2}{2} + a_3 \frac{x^3}{3} + a_4 \frac{x^4}{4}\right]_{-1}^{1} - W_1 f(x_1) - W_2 f(x_2) = 0$$

$$\left[a_1 x + a_2 \frac{x^2}{2} + a_3 \frac{x^3}{3} + a_4 \frac{x^4}{4}\right]_{-1}^{1} - W_1 f(x_1) - W_2 f(x_2) = 0$$

$$\left[2a_1 + 0 + \frac{2}{3} a_3 + 0\right] - W_1\left(a_1 + a_2 x_1 + a_3 x_1^2 + a_3 x_1^3\right) - W_2\ \left(a_1 + a_2 x_2 + a_3 x_2^2 + a_3 x_2^3\right) = 0$$

$$a_1[2 - (W_1 + W_2)] - a_2[W_1 x_1 + W_2 x_2] - a_3\left[W_1 x_1^2 + W_2 x_2^2 - \frac{2}{3}\right] - a_4[W_1 x_1^3 + W_2 x_2^3] = 0$$

satisfies only if,

$$W_1 + W_2 = 2$$

$$W_1 x_2 + W_2 x_2 = 0$$

$$W_1 x_1^2 + W_2 x_2^2 = \frac{2}{3}$$

$$W_1 x_1^3 + W_2 x_2^3 = 0$$

On solving the above equations,

$$W_1 = W_2 = 1$$

$$x_1 = \frac{1}{\sqrt{3}} = 0.5773502$$

$$x_2 = \frac{-1}{\sqrt{3}} = -0.5773502$$

On solving the above equations,

$$W_1 = W_2 = 1$$

$$x_1 = \frac{1}{\sqrt{3}} = 0.5773502$$

$$x_2 = \frac{-1}{\sqrt{3}} = -0.5773502$$

$$I = \int_{-1}^{1} f(x)dx = (1)f(0.5773502) + (1)f(-0.5773502)$$

Let, $n = 2$.

$$I = \int_{-1}^{1} f(x)dx = W_1 f(x_1) + W_2 f(x_2)$$

Two Dimensional Problem

Integral to evaluate two dimensional problem,

$$I = \int_{-1}^{1} \int_{-1}^{1} f(x,y)\, dx\, dy$$

On generalizing the relation,

$$I \simeq \int_{-1}^{1} \left[\sum_{i=1}^{n} W_i f(x_i, y) \right] dy$$

$$\simeq \sum_{j=1}^{n} W_j \left[\sum_{i=1}^{n} W_i f(x_i, y_j) \right]$$

$$I \simeq \sum_{i=1}^{n} \sum_{j=1}^{n} W_i W_j\, f\left(x_i, y_j\right)$$

Using two Gauss integration points,

For $n = 2$,

$$x_i = y_j = \pm\, 0.57735$$

$$W_i = 1$$

i.e.,

$$x_1 = y_1 = 0.5773j$$

$$x_2 = y_2 = -\, 0.57735$$

$$W_1 = W_2 = 1$$

Then, equation (2) becomes,

$$I \simeq W_1^2 f(x_1, y_1) + W_2^2 f(x_2, y_2)$$

$$\therefore\ I \simeq f(\,0.57735,\ 0.57735) + f(-\,0.57735, -\,0.57735)$$

Unit 4

Dynamic Analysis:

Formulation of

finite element model,

element consistent and lumped mass matrices,

Evaluation of eigenvalues and eigenvectors,

Free vibration analysis.

Steady state heat transfer analysis:

one dimensional analysis of a fin.

Introduction to FE software.

Types of analysis of a Problem

1. Static Analysis

In case, if the interia effects are not considered and damping is zero, then it is said to be quasistatic and the analysis is a static analysis.

$$[K]\{\delta\} = \{F\}$$

Where,

$[K]$ – Global stiffness matrix

$\{\delta\}$ – Global displacement vector

$\{F\}$ – Global load vector

Types of analysis of a Problem

2. Eigen Value Analysis

If the interia effects are taken into consideration with zero damping and applied loads, then the equations of motion is reduced to a generalised eigne value problem. In such cases, following can be used to obtain the solution,

$$[M]\{\ddot{\delta}\} + [K]\{\delta\} = 0$$

Where,

$[M]$ – Global mass matrix

$\{\ddot{\delta}\}$ – Global nodal acceleration vector

Types of analysis of a Problem

3. Transient Dynamic Analysis

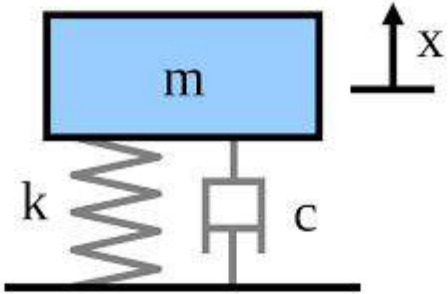
If the loads are arbitrary but known functions of time, then the analysis is known as transient dynamic analysis. In such cases,

$$[M]\{\ddot{\delta}\}+[C]\{\dot{\delta}\}+[K]\{\delta\}=F(t)$$

Where,

$[C]$ – Global viscous damping matrix

$\{\dot{\delta}\}$ – Global nodal velocity vector



4. Frequency Response Analysis

If the structure is subjected to harmonic loads, then the analysis is known as frequency response analysis. In such cases,

$$[M]\{\ddot{\delta}\}+[C]\{\dot{\delta}\}+[K]\{\delta\}= F\sin(\omega t)$$

Steps in an FEM Analysis

Preprocessor/Modeling

Analysis run / Solve

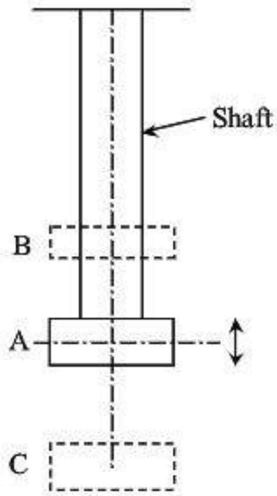
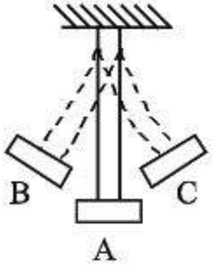
Post-processing/View results

Analysis run / Solve:

- Formulation of element stiffness matrices
- Assembly of global stiffness matrix [K]
- formulation of load vector {F}
- Solution of [K]{u}={F} to get nodal displacements {u}

in case of dynamic analysis)

$$[K] \Rightarrow [K] - \lambda[M]$$

	Longitudinal Vibrations		Transverse Vibrations
1.	The vibrations, in which the particles of the shaft move parallel to the axis of the shaft are called longitudinal vibrations.	1.	The vibrations, in which particles of the shaft move perpendicular to the axis of the shaft are called transverse vibrations.
2.	 Figure: Longitudinal Vibration	2.	 Figure: Transverse Vibration

	Longitudinal Vibrations		Transverse Vibrations
3.	In case of longitudinal vibrations, the shaft is elongated and shortened alternately.	3.	In case of transverse vibrations, the shaft is straight and bent alternately.
4.	Tensile and compressive stresses are induced alternately.	4.	Bending stresses are induced.
5.	The natural frequency of free longitudinal vibrations is, $f_n = \frac{0.4985}{\sqrt{\delta}}$	5.	The natural frequency of transverse vibrations is, $f_n = \frac{0.4985}{\sqrt{\delta}}$

$$f_{n1} = \frac{1}{2\pi}\sqrt{\frac{g}{\Delta_1}} = \frac{\sqrt{9.81}}{2\pi}\frac{1}{\sqrt{\Delta_1}} = \frac{0.4985}{\sqrt{\Delta_1}}$$

longitudinal vibration of bar element,

Finite Element Equation

$$[[K]-\omega^2[M]]\{\delta\} = \{F\}$$

$[K]$ – Stiffness matrix

$[M]$ – Mass matrix

$\{\delta\}$ – Displacement vector

$\{F\}$ – Force element

Stiffness Matrix

$$[K] = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

E – Young's modulus of the bar material

A – Area of crossection of the bar

L – Length of the bar

transverse vibration of beam element,

Finite Element Equation

$$[[K]-\omega^2[M]]\{\delta\} = \{F\}$$

Stiffness Matrix

$$[K] = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$$

longitudinal vibration of bar element,

Consistent Mass Matrix

$$[M] = \frac{\rho AL}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

ρ – Density of the bar element

A – Area of cross-section of bar

L – Length of the bar

Lumped Mass Matrix

$$[M] = \frac{\rho AL}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

transverse vibration of beam element,

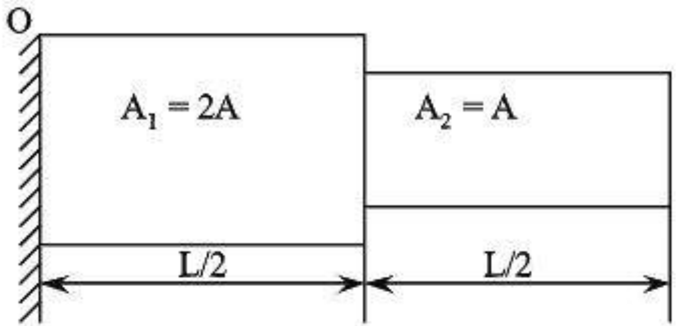
Consistent Mass Matrix

$$[M] = \frac{\rho AL}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix}$$

Lumped Mass Matrix

$$[M] = \frac{\rho Al}{2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Find the natural frequencies for the longitudinal vibrations of the stepped bar. Assume $A_1 = 2A$, $A_2 = A$ and $E_1 = E_2 = E$.



Figure(1)

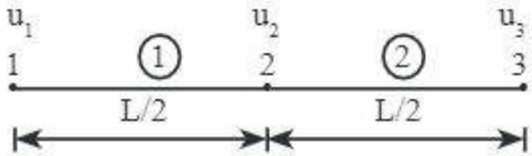


Figure (2): Bar Element Model

$$[[K] - \omega^2[M]] \{\delta\} = \{F\}$$

Stiffness Matrix

For element (1),

$$\therefore \quad [K_1] = \frac{E_1 A_1}{L_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{E(2A)}{L/2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[K_1] = \frac{2EA}{L}\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$$

For element (2),

$$[K_2] = \frac{E_2 A_2}{L_2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{E(A)}{L/2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{2EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Global Stiffness Matrix

$$\therefore \quad [K] = [K_1] + [K_2]$$

$$[K] = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix} \times \frac{2EA}{L}$$

Mass Matrix

For element (1),

$$[M_1] = \frac{\rho . A_1 L_1}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= \frac{\rho (2A)(L/2)}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$[M_1] = \frac{\rho AL}{12}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$$

For element (2),

$$[M_2] = \frac{\rho . A_2 L_2}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= \frac{\rho (A)(L/2)}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$[M_2] = \frac{\rho AL}{12}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Global mass matrix,

$$[M] = [M_1] + [M_2]$$

$$[M] = \begin{bmatrix} 4 & 2 & 0 \\ 2 & 6 & 1 \\ 0 & 1 & 2 \end{bmatrix} \times \frac{\rho AL}{12}$$

Finite element equation for the longitudinal vibrations of a bar element is given by,

$$[[K] - \omega^2[M]]\ \{\delta\} = \{F\}$$

$$\left[\frac{2AE}{L}\begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix} - \omega^2 . \frac{\rho AL}{12}\begin{bmatrix} 4 & 2 & 0 \\ 2 & 6 & 1 \\ 0 & 1 & 2 \end{bmatrix}\right]\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

Applying the boundary conditions,

$$F_1 = 0, F_2 = 0. F_3 = 0$$

$$\left[\frac{2AE}{L}\begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix} - \omega^2 . \frac{\rho AL}{12}\begin{bmatrix} 4 & 2 & 0 \\ 2 & 6 & 1 \\ 0 & 1 & 2 \end{bmatrix}\right]\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = 0$$

****One can eliminate the 1st row and column to reduce computation**

Characteristic equation is given by,

$$|[K] - \omega^2[M]| = 0$$

$$\left| \frac{2AE}{L}\begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix} - \omega^2 \cdot \frac{\rho AL}{12}\begin{bmatrix} 4 & 2 & 0 \\ 2 & 6 & 1 \\ 0 & 1 & 2 \end{bmatrix} \right| = 0$$

Divide by $\dfrac{2AE}{L}$ on both sides.

$$\left| \begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix} - \frac{\frac{\rho AL}{12}}{\frac{2AE}{L}}\omega^2 \begin{bmatrix} 4 & 2 & 0 \\ 2 & 6 & 1 \\ 0 & 1 & 2 \end{bmatrix} \right| = 0$$

$$\left| \begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix} - \frac{\rho L^2 \omega^2}{24E}\omega^2 \begin{bmatrix} 4 & 2 & 0 \\ 2 & 6 & 1 \\ 0 & 1 & 2 \end{bmatrix} \right| = 0$$

Let,

$$\alpha = \frac{\rho L^2 \omega^2}{24E}$$

$$\left| \begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix} - \alpha \begin{bmatrix} 4 & 2 & 0 \\ 2 & 6 & 1 \\ 0 & 1 & 2 \end{bmatrix} \right| = 0$$

$$\left| \begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix} - \begin{bmatrix} 4\alpha & 2\alpha & 0 \\ 2\alpha & 6\alpha & \alpha \\ 0 & \alpha & 2\alpha \end{bmatrix} \right| = 0$$

$$\left|\begin{bmatrix} (2-4\alpha) & (-2-2\alpha) & 0 \\ (-2-2\alpha) & (3-6\alpha) & (-1-\alpha) \\ 0 & (-1-\alpha) & (1-2\alpha) \end{bmatrix}\right| = 0$$

$$(2-4\alpha)[(3-6\alpha)(1-2\alpha) - (-1-\alpha)(-1-\alpha)]$$

$$-(-2-2\alpha)[(2-4\alpha)(1-2\alpha)] = 0$$

$$(2-4\alpha)[3-6\alpha-6\alpha+12\alpha^2-1-\alpha-\alpha-\alpha^2]$$

$$-(-2-2\alpha)[(2-4\alpha-4\alpha+8\alpha^2)] = 0$$

$$18\alpha(1-2\alpha)(\alpha-2) = 0$$

$$\alpha = 0, \frac{1}{2}, 2$$

The roots of above equation gives the natural frequencies of the bar,

When,

$$\alpha = 0$$

$$\alpha = \frac{\rho L^2 \omega_1^2}{2E}$$

$$\omega_1^2 = 0$$

$$\omega_1 = 0 \text{ rad/sec}$$

When,

$$\alpha = \frac{1}{2}$$

$$\alpha = \frac{\rho L^2 \omega_2^2}{24E}$$

$$\frac{1}{2} = \frac{\rho L^2 \omega_2^2}{24E}$$

$$\omega_2^2 = \frac{12E}{\rho L^2}$$

$$\omega_2 = \sqrt{\frac{12E}{\rho L^2}}$$

$$\omega_2 = 3.464\sqrt{\frac{E}{\rho L^2}} \text{ rad/sec}$$

When,

$$\alpha = 2$$

$$\alpha = \frac{\rho L^2 \omega_3^2}{24E}$$

$$2 = \frac{\rho L^2 \omega_3^2}{24E}$$

$$\omega_3^2 = \frac{48E}{\rho L^2}$$

$$\omega_3 = \sqrt{\frac{48E}{\rho L^2}}$$

$$\omega_3 = 6.928\sqrt{\frac{E}{\rho L^2}} \text{ rad/sec}$$

Therefore, the natural frequencies of longitudinal vibrations of given stepped bar is given by,

$$\omega_1 = 0 \text{ rad/sec}$$

$$\omega_2 = 3.464\sqrt{\frac{E}{\rho L^2}} \text{ rad/sec}$$

$$\omega_3 = 6.928\sqrt{\frac{E}{\rho L^2}} \text{ rad/sec}$$

Equations of Motion Using Lagrange's Approach

We define the Lagrangean by

$$L = T - \Pi$$

If T represents the kinetic energy of a system and π represents potential energy, then Lagrange's equations of motion

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0} \longrightarrow \boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q} + \frac{\partial \pi}{\partial q} = F\Big|_q}$$

where $F|_q$ represents the generalised force in the coordinate q and $\dot{q} = dq/dt$.

Lagrange's equations of motion, we obtain

$$[M]\{\ddot{\delta}\} + [K]\{\delta\} - \{F\} = \{0\}$$

$$\boxed{[M]\{\ddot{\delta}\} + [K]\{\delta\} = \{F\}}$$

Solid Body with Distributed Mass

$$\dot{\mathbf{u}} = [\dot{u} \quad \dot{v} \quad \dot{w}]^{\mathrm{T}}$$

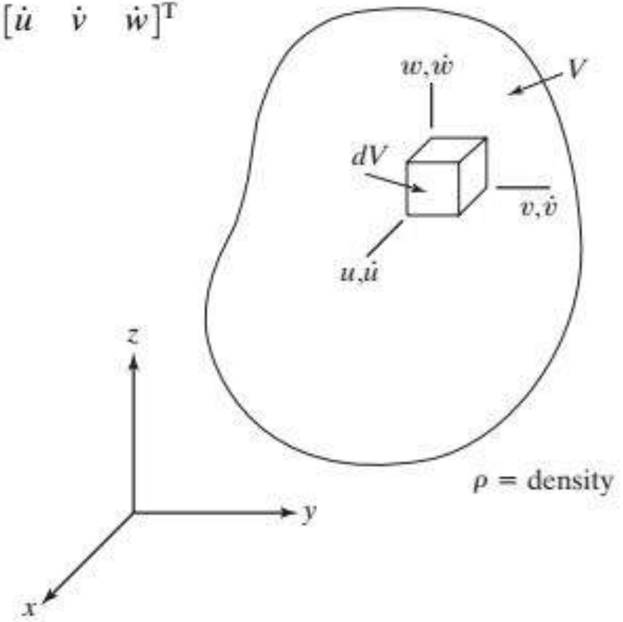


FIGURE 11.1 Body with distributed mass.

we express
displacement **u** in terms of
the nodal displacements **q**,
using shape functions **N**

$$\mathbf{u} = \mathbf{N}\mathbf{q}$$

In dynamic analysis,
the elements of **q** are dependent on time,
while **N** represents (spatial) shape functions.

The velocity vector is then given by $\dot{\mathbf{u}} = \mathbf{N}\dot{\mathbf{q}}$

The kinetic energy $T = \dfrac{1}{2}\displaystyle\int_V \dot{\mathbf{u}}^{\mathrm{T}}\dot{\mathbf{u}}\rho\, dV$

$$T_e = \frac{1}{2}\dot{\mathbf{q}}^{\mathrm{T}}\left[\int_e \rho \mathbf{N}^{\mathrm{T}}\mathbf{N}\, dV\right]\dot{\mathbf{q}}$$

$$\boxed{\mathbf{m}^e = \int_e \rho \mathbf{N}^{\mathrm{T}}\mathbf{N}\, dV}$$

This mass matrix is consistent with the shape functions chosen and is called the consistent mass matrix.

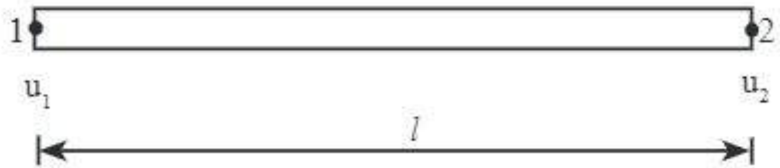


Figure: Bar Element

The above figure shows a bar element of length 'l'. Let u_1, u_2 are the nodal displacements at nodal points 1,2.

$$N_1 = \frac{l-x}{l}, \; N_2 = \frac{x}{l}$$

Shape functions for a $1-D$ bar element with two nodes is given by,

$$\boxed{\mathbf{m}^e = \int_e \rho \mathbf{N}^{\mathrm{T}} \mathbf{N} \, dV}$$

Mass matrix,

$$[M] = \int_v \rho.[N]^T[N]dV \qquad = \rho \int_0^l \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \begin{bmatrix} N_1 & N_2 \end{bmatrix} A.dx$$

$$= \rho A \int_0^l \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \begin{bmatrix} N_1 & N_2 \end{bmatrix} .dx$$

$$= \rho A \int_0^l \begin{bmatrix} \frac{l-x}{l} \\ \frac{x}{l} \end{bmatrix} \begin{bmatrix} \frac{l-x}{l} & \frac{x}{l} \end{bmatrix} dx$$

$$= \rho A \int_0^l \begin{bmatrix} \left(\frac{l-x}{l}\right)^2 & \frac{x}{l} - \frac{x^2}{l^2} \\ \frac{x}{l} - \frac{x^2}{l^2} & \frac{x^2}{l^2} \end{bmatrix} dx$$

$$= \rho A \begin{bmatrix} \frac{\left(1-\frac{x}{l}\right)^2}{3\left(\frac{-1}{l}\right)} & \frac{x^2}{2l} - \frac{x^3}{3l^2} \\ \frac{x^2}{2l} - \frac{x^3}{3l^2} & \frac{x^3}{3l^2} \end{bmatrix}_0^l$$

$$= \rho A \begin{bmatrix} \frac{\left(1-\frac{l}{l}\right)^2}{3\left(\frac{-1}{l}\right)} & \frac{l^2}{2l} - \frac{l^3}{3l^2} \\ \frac{l^2}{2l} - \frac{l^3}{3l^2} & \frac{l^3}{3l^2} \end{bmatrix}$$

$$\boxed{= \rho A \begin{bmatrix} \frac{l}{3} & \frac{l}{6} \\ \frac{l}{6} & \frac{l}{3} \end{bmatrix} = \frac{\rho A l}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}}$$

Bar element $\boxed{N_1 = \dfrac{l-x}{l},\ N_2 = \dfrac{x}{l}}$

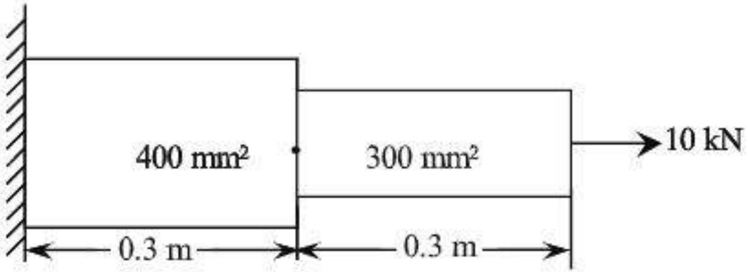
$$[m]^e = \int_v \rho [N]^T [N]\, dv = \rho A \int_0^\ell \begin{bmatrix} 1 - \dfrac{x}{\ell} \\ \dfrac{x}{\ell} \end{bmatrix} \left[\left(1 - \frac{x}{\ell}\right)\left(\frac{x}{\ell}\right)\right] dx = \frac{\rho A \ell}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Beam element

$$\boxed{\begin{aligned} &N_1 = 1 - 3x^2/\ell^2 + 2x^3/\ell^3, \quad N_2 = x - 2x^2/\ell + x^3/\ell^2 \\ &N_3 = 3x^2/\ell^2 - 2x^3/\ell^3, \quad N_4 = -x^2/\ell + x^3/\ell^2 \end{aligned}}$$

$$[m]^e = \int_v \rho [N]^T [N]\, dv = \rho A \int_0^\ell \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} \begin{bmatrix} N_1 & N_2 & N_3 & N_4 \end{bmatrix} dx = \frac{\rho A \ell}{420} \begin{bmatrix} 156 & & & \text{Symmetric} \\ 22\ell & 4\ell^2 & & \\ 54 & 13\ell & 156 & \\ -13\ell & -3\ell^2 & -22\ell & 4\ell^2 \end{bmatrix}$$

Determine the natural frequencies and mode shapes of a stepped bar as shown in figure, using the characteristic polynomial technique. Assume E = 250 GPa and density is 7850 kg/m³.



Figure

Determine the natural frequencies of a stepped bar as shown in the figure. (E=250 GPa, density 7850 kg/m^3

Given that,

Young's modulus, $E = 250$ GPa $= 250 \times 10^9$ Pa

Density, $\rho = 7850$ kg/m^3

Evaluation is done using characteristic polynomial technique.

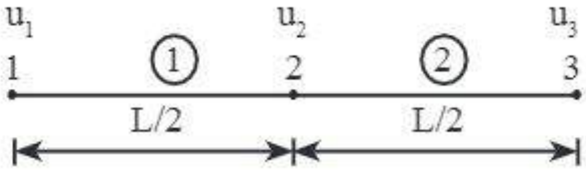


Figure (1): Bar Element Model

Finite element equation of a bar element is given by,

$$\left[[K]-\omega^2[M]\right]\{\delta\}=\{F\}$$

Stiffness Matrix

For element (1),

$$[K_1]=\frac{EA_1}{L_1}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

$$=\frac{250\times10^9\times400\times10^{-6}}{0.3}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

$$=333.333\times10^6\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

$$[K_1]=10^6\begin{array}{c c} 1 & 2 & \\ \left[\begin{matrix}333.333 & -333.333\\ -333.333 & 333.333\end{matrix}\right] & \begin{matrix}1\\2\end{matrix}\end{array}$$

For element (2),

$$[K_2]=\frac{EA_2}{L_2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

$$=\frac{250\times10^9\times300\times10^{-6}}{0.3}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

$$[K_2]=10^6\begin{array}{cc} 2 \quad\quad 3 & \\ \left[\begin{matrix}250 & -250\\ -250 & 250\end{matrix}\right] & \begin{matrix}2\\3\end{matrix}\end{array}$$

Global stiffness matrix,

$$\therefore \quad [K]=[K_1]+[K_2]$$

$$[K]=10^6\begin{array}{cc} 1 \quad\quad\quad 2 \quad\quad\quad 3 & \\ \left[\begin{matrix}333.333 & -333.333 & 0\\ -333.333 & 583.333 & -250\\ 0 & -250 & 250\end{matrix}\right] & \begin{matrix}1\\2\\3\end{matrix}\end{array}$$

Mass Matrix

For element (1),

$$[M_1] = \frac{\rho A_1 L_1}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= \frac{7850 \times 400 \times 10^{-6} \times 0.3}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= 0.157\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$[M_1] = \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 0.314 & 0.157 \\ 0.157 & 0.314 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \end{array}$$

For element (2),

$$[M_2] = \frac{\rho A_2 l_2}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= \frac{7850 \times 300 \times 10^{-6} \times 0.3}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= 0.117\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$[M_2] = \begin{array}{c} \begin{array}{cc} 2 & 3 \end{array} \\ \begin{bmatrix} 0.234 & 0.117 \\ 0.117 & 0.234 \end{bmatrix} \end{array} \begin{array}{c} \\ 2 \\ 3 \end{array}$$

Global mass matrix,

$$\because \quad [M] = [M_1] + [M_2]$$

$$\therefore \quad [M] = \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.314 & 0.157 & 0 \\ 0.157 & 0.548 & 0.117 \\ 0 & 0.117 & 0.234 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}$$

$$[[K]-\omega^2[M]]\ \{u\}=\{F\}$$

$$\left[10^6\begin{bmatrix}333.333 & -333.333 & 0\\ -333.333 & 583.333 & -250\\ 0 & -250 & 250\end{bmatrix}-\omega^2\begin{bmatrix}0.314 & 0.157 & 0\\ 0.157 & 0.548 & 0.117\\ 0 & 0.117 & 0.234\end{bmatrix}\right]\begin{Bmatrix}u_1\\ u_2\\ u_3\end{Bmatrix}=\begin{Bmatrix}0\\ 0\\ 10^4\end{Bmatrix} \longrightarrow \textbf{Eq-1}$$

Apply the boundary conditions to the above equation,

$$u_1=0$$

Eliminating first row and first column of both the matrixes,

$$\left[10^6\begin{bmatrix}583.333 & -250\\ -250 & 250\end{bmatrix}-\lambda\begin{bmatrix}0.548 & 0.117\\ 0.117 & 0.234\end{bmatrix}\right]\begin{Bmatrix}u_2\\ u_3\end{Bmatrix}=\begin{Bmatrix}0\\ 10^4\end{Bmatrix} \longrightarrow \textbf{Eq-2}$$

Let, $\lambda=\omega^2$.

Characteristic equation is given by,

$$|[K]-\lambda[M]|=0$$

$$\left|10^6\begin{bmatrix}583.333 & -250\\ -250 & 250\end{bmatrix}-\lambda\begin{bmatrix}0.548 & 0.117\\ 0.117 & 0.234\end{bmatrix}\right|=0$$

$$\left| 10^6 \begin{bmatrix} 583.333 & -250 \\ -250 & 250 \end{bmatrix} - \begin{bmatrix} 0.548\lambda & 0.117\lambda \\ 0.117\lambda & 0.234\lambda \end{bmatrix} \right| = 0$$

$$\left| \begin{bmatrix} 583.333\times 10^6 - 0.548\lambda & -250\times 10^6 - 0.117\lambda \\ -250\times 10^6 - 0.117\lambda & 250\times 10^6 - 0.234\lambda \end{bmatrix} \right| = 0$$

$$\left[\left(583.333\times 10^6 - 0.548\lambda\right)\left(250\times 10^6 - 0.234\lambda\right) - \left(-250\times 10^6 - 0.117\lambda\right)\left(-250\times 10^6 - 0.117\lambda\right) \right] = 0$$

$$\left(583.333\times 10^6 \times 250\times 10^6\right) - \left(583.333\times 10^6 \times 0.234\lambda\right) - \left(0.548\lambda \times 250\times 10^6\right) + (0.548\lambda \times 0.234\lambda)$$

$$-\left(250\times 10^6 \times 250\times 10^6\right) - \left(250\times 10^6 \times 0.117\lambda\right) - \left(0.117\lambda \times 250\times 10^6\right) - (0.117\lambda \times 0.117\lambda) = 0$$

$$1.45 \times 10^{17} - 136.4 \times 10^6\lambda - 137 \times 10^6\lambda + 0.128\,\lambda^2 - 6.25 \times 10^{16} - 29.25 \times 10^6\lambda - 29.25 \times 10^6\lambda - 0.0136\lambda^2 = 0$$

$$\boxed{0.115\lambda^2 - 331.9 \times 10^6\,\lambda + 8.25 \times 10^{16} = 0}$$

$$\lambda_1 = 2611368423$$

$$\lambda_2 = 274718533$$

$\lambda_1 = 2611368423$

$\lambda_1 = \omega_1^2 = 2611368423$

$\omega_1 = \sqrt{2611368423}$

$\omega_1 = 51101.55$ rad/sec.

Natural frequency of element (1) is, $f_1 = \dfrac{\omega_1}{2\pi}$

$$f_1 = \frac{51101.55}{2\pi}$$

$$f_1 = 8133.064 \text{ Hz}$$

When,

$\lambda_2 = 274718533$

$\lambda_2 = \omega_2^2 = 274718533$

$\omega_2 = \sqrt{274718533}$

$\omega_2 = 16574.63$ rad/sec

Natural frequency of element (2),

$$f_2 = \frac{\omega_2}{2\pi}$$

$$= \frac{16574.63}{2\pi}$$

$$f_2 = 2637.93 \text{ Hz}$$

Mode Shapes

From equation (2)

$$\left[10^6\begin{bmatrix}583.33 & -250\\ -250 & 250\end{bmatrix} - \lambda\begin{bmatrix}0.548 & 0.117\\ 0.117 & 0.237\end{bmatrix}\right]\begin{Bmatrix}u_2\\ u_3\end{Bmatrix} = \begin{Bmatrix}0\\ 10^4\end{Bmatrix}$$

$$\begin{bmatrix}583.33\times10^6 - 0.548\,\lambda & -250\times10^6 - 0.117\lambda\\ -250\times10^6 - 0.117\lambda & 250\times10^6 - 0.234\lambda\end{bmatrix}\begin{Bmatrix}u_2\\ u_3\end{Bmatrix} = \begin{Bmatrix}0\\ 10^4\end{Bmatrix}$$

For $\lambda = \lambda_1$

$$\lambda_1 = 2611368423$$

$$\begin{Bmatrix}583.33\times10^6 - 0.548(2611368423) & -250\times10^6 - 0.117(2611368423)\\ -250\times10^6 - 0.117(2611368423) & 250\times10^6 - 0.234(2611368423)\end{Bmatrix}\begin{Bmatrix}u_2\\ u_3\end{Bmatrix} = \begin{Bmatrix}0\\ 10^4\end{Bmatrix}$$

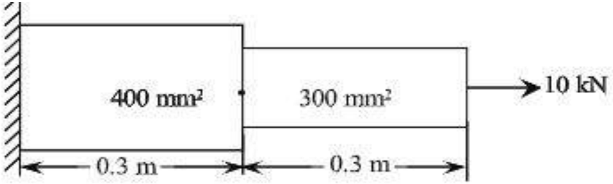
$$\begin{bmatrix}-847699895.8 & -555530105.5\\ -555530105.5 & -361060211\end{bmatrix}\begin{Bmatrix}u_2\\ u_3\end{Bmatrix} = \begin{Bmatrix}0\\ 10^4\end{Bmatrix}$$

$$\begin{bmatrix} -847699895.8 & -555530105.5 \\ -555530105.5 & -361060211 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 10^4 \end{Bmatrix}$$

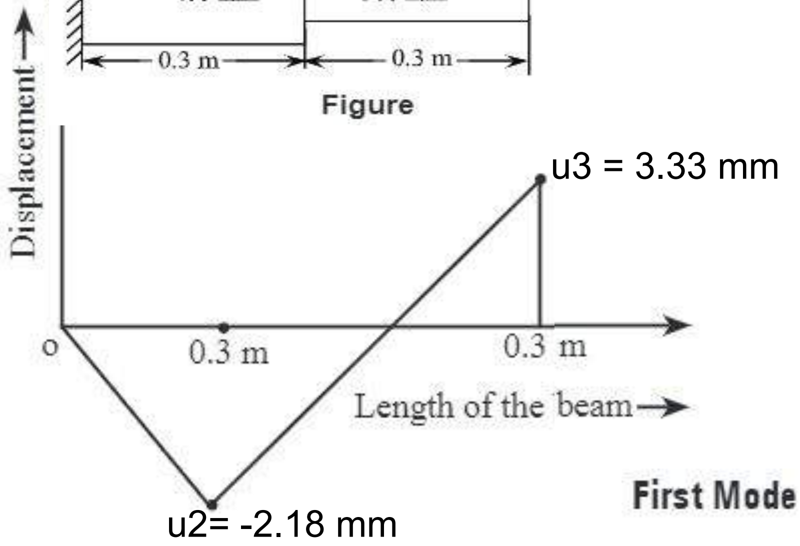
$$\begin{pmatrix} -84769 & -55553 \\ -55553 & -36106 \end{pmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

	u2	u3
solution	**-0.00218**	**0.00333**

1st Mode shape



Figure



First Mode

Mode Shapes

For second mode shape

From equation (2)

$$\left[10^6\begin{bmatrix}583.33 & -250\\ -250 & 250\end{bmatrix} - \lambda\begin{bmatrix}0.548 & 0.117\\ 0.117 & 0.237\end{bmatrix}\right]\begin{Bmatrix}u_2\\ u_3\end{Bmatrix} = \begin{Bmatrix}0\\ 10^4\end{Bmatrix}$$

$$\begin{bmatrix}583.33\times10^6 - 0.548\,\lambda & -250\times10^6 - 0.117\lambda\\ -250\times10^6 - 0.117\lambda & 250\times10^6 - 0.234\lambda\end{bmatrix}\begin{Bmatrix}u_2\\ u_3\end{Bmatrix} = \begin{Bmatrix}0\\ 10^4\end{Bmatrix}$$

For $\lambda = \lambda_2$

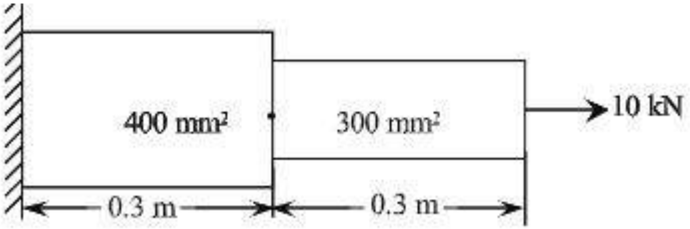
$$\lambda_2 = 274718533$$

$$\begin{bmatrix}583.33\times10^6 - 0.548\,(274718533) & -250\times10^6 - 0.117(274718533)\\ -250\times10^6 - 0.117(274718533) & 250\times10^6 - 0.234(274718533)\end{bmatrix}\begin{Bmatrix}u_2\\ u_3\end{Bmatrix} = \begin{Bmatrix}0\\ 10^4\end{Bmatrix}$$

$$\begin{bmatrix}432784243.9 & -282142068.4\\ -282142068.4 & 185715863.3\end{bmatrix}\begin{Bmatrix}u_2\\ u_3\end{Bmatrix} = \begin{Bmatrix}0\\ 10^4\end{Bmatrix}$$

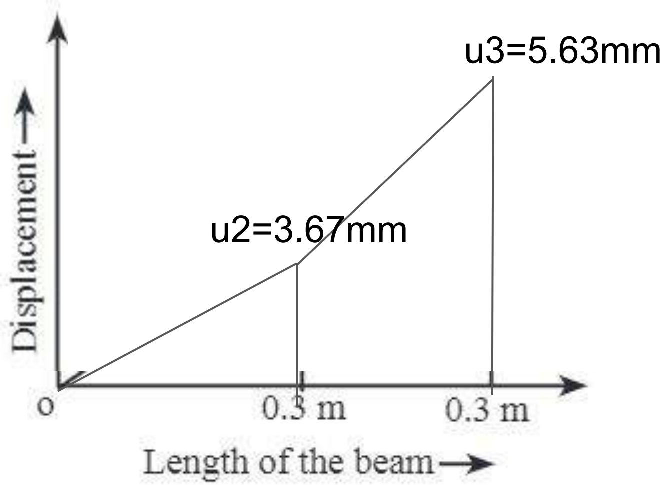
$$\begin{pmatrix} 43278 & -28214 \\ -28214 & 18571 \end{pmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

	u2	u3
solution	0.00367	0.00563



Figure

Second Mode



$$\mathbf{KU} = \omega^2\mathbf{MU}$$

$$[\,K-\lambda\, M\,]\,\{U\} = \{\,0\,\}$$

This is the generalized eigenvalue problem

$$\mathbf{KU} = \lambda\mathbf{MU}$$

where **U** is the eigenvector, representing the vibrating mode, corresponding to the eigenvalue λ. The eigenvalue λ is the square of the circular frequency ω. The frequency f in hertz (cycles per second) is obtained from $f = \omega/(2\pi)$.

EVALUATION OF EIGENVALUES AND EIGENVECTORS

The generalized problem in free vibration is that of evaluating an eigenvalue $\lambda(=\omega^2)$,

which is a measure of the frequency of vibration together with the corresponding eigenvector **U** indicating the mode shape, as in

$$\mathbf{KU} = \lambda\mathbf{MU}$$

$$[\,K-\lambda\, M\,]\,\{U\} = \{\,0\,\}$$

λ	-------- **Eigen Value (Frequency)**
U or u	-------- **Eigen Vector (Mode shape)**

Eigenvalue–Eigenvector Evaluation

The eigenvalue–eigenvector evaluation procedures fall into the following basic categories:

1. Characteristic polynomial technique
2. Vector iteration methods
3. Transformation methods

Characteristic Polynomial. From Eq. 11.38, we have

$$(\mathbf{K} - \lambda\mathbf{M})\mathbf{U} = \mathbf{0}$$

If the eigenvector is to be nontrivial, the required condition is

$$\det(\mathbf{K} - \lambda\mathbf{M}) = \mathbf{0}$$

This represents the characteristic polynomial in λ.

What is study state heat transfer analysis? Write its governing Equation?

Steady state heat transfer is defined as the temperature at any point in the medium does not change with time.

For a one dimensional steady state heat transfer,

$$K.\frac{d^2T}{dx^2} + q = 0$$

K – Thermal conductivity

T – Temperature

q – Internal heat source per unit volume

Steady State Conduction	
1.	In steady state conduction, the temperature at any point in the medium does not change with time.
2.	It is due to the rate of heat conducted into the medium is equal to the rate of heat conducted out of the medium.
3.	It is a function of space coordinates and is given by, $T = T(x, y, z)$
4.	One dimensional heat conduction is given by, $\frac{\partial}{\partial x}\left[k\frac{\partial T}{\partial x}\right] + q = 0$

3D Conduction heat transfer

General 3D conduction Equation:

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \dot{q} = \rho c\frac{\partial T}{\partial \tau}$$

For constant conductivity:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial \tau}$$

$$\alpha = k/\rho c$$

= **Thermal diffusivity of a material**

Q. **Give the finite element equation for a one dimensional heat conduction element.**

Ans: The finite element equation for a one dimensional heat conduction element is given by,

$$\{F\} = [K_c]\{T\}$$

$\{F\}$ – Force vector

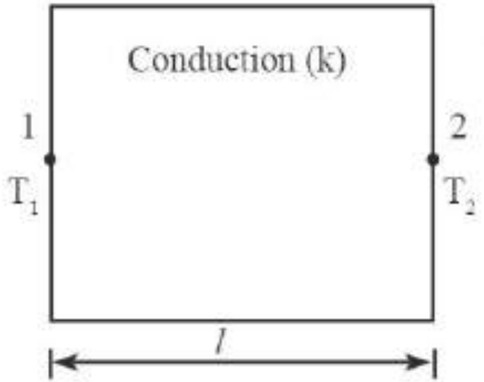
$$= \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \text{ for a two noded element}$$

$[K_c]$ – Stiffness matrix in case of heat conduction

$$= \frac{KA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$\{T\}$ – Nodal temperature vector

$$= \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} \text{ for a two noded element}$$



Similar to structural problems

- Formulation of element stiffness matrices
- Assembly of global stiffness matrix [K]
- formulation of load vector {F}
- Solution of [K]{u}={F} to get nodal displacements {u}

Heat Transfer Rate Due to Conduction

$$Q = -kA.\frac{\Delta T}{L} \text{ Watts}$$

Where,

k – Thermal conductivity

A – Surface area

ΔT – Temperature difference

L – Length.

Heat Transfer Rate Due to Convection

$$Q = hA(T_s - T_\infty) \text{ Watts}$$

Where,

h – Heat transfer coefficient

T_s – Surface temperature

T_∞ – Ambient temperature.

Heat Transfer Rate Due to Radiation

$$Q = \sigma A(\Delta T)^4 \text{ Watts}$$

Where,

σ – Stefan-Boltzmann constant

$\sigma = 5.67 \times 10^{-8}$ W/m^2 K^4.

Derive the stiffness matrix for a one dimensional heat conduction element.

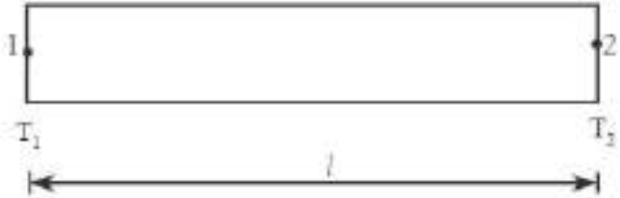


Figure: Bar element

The above figure shows a bar element of length 'l'. Let T_1, T_2 are the temperature at nodes 1,2 and 'K' be the thermal conductivity of the bar element.

Stiffness matrix,

$$[K] = \int_v [B]^T [D][B] dV \qquad \dots (1)$$

[B] – Strain displacement matrix

[D] – Stress – strain matrix

For a one dimensional bar element,

Temperature function, $T = N_1 T_1 + N_2 T_2$

Where,

$$N_1 = \frac{l-x}{l}$$

$$N_2 = \frac{x}{l}$$

Strain-Displacement matrix,

$$[B] = \begin{bmatrix} \frac{dN_1}{dx} & \frac{dN_2}{dx} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{d}{dx}\left[\frac{l-x}{l}\right] & \frac{d}{dx}\left(\frac{x}{l}\right) \end{bmatrix} = \begin{bmatrix} \frac{-1}{l} & \frac{1}{l} \end{bmatrix}$$

$$[B]^T = \begin{bmatrix} \frac{-1}{l} \\ \frac{1}{l} \end{bmatrix}$$

For a one dimensional heat conduction,

$$[D] = K$$

From equation (1),

$$[K] = \int_v [B]^T [D][B] dV$$

$$[K_c] = \int_0^l \begin{bmatrix} \frac{-1}{l} \\ \frac{1}{l} \end{bmatrix} .K. \begin{bmatrix} \frac{-1}{l} & \frac{1}{l} \end{bmatrix} dV$$

$[K_c]$ – Stiffness matrix in case of conduction

$$= \int_0^l \begin{bmatrix} \frac{1}{l^2} & \frac{-1}{l^2} \\ \frac{-1}{l^2} & \frac{1}{l^2} \end{bmatrix} K.dV$$

$$= \int_0^l \begin{bmatrix} \frac{1}{l^2} & \frac{-1}{l^2} \\ \frac{-1}{l^2} & \frac{1}{l^2} \end{bmatrix} K.A.dx$$

$$= KA \begin{bmatrix} \frac{1}{l^2} & \frac{-1}{l^2} \\ \frac{-1}{l^2} & \frac{1}{l^2} \end{bmatrix} \int_0^l 1.dx \qquad = \frac{KA}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

FE Equation for 1D heat conduction element

The finite element equation for a one dimensional heat conduction element is given by,

$$\{F\} = [K_c]\{T\}$$

$\{F\}$ – Force vector

$$= \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \text{ for a two noded element}$$

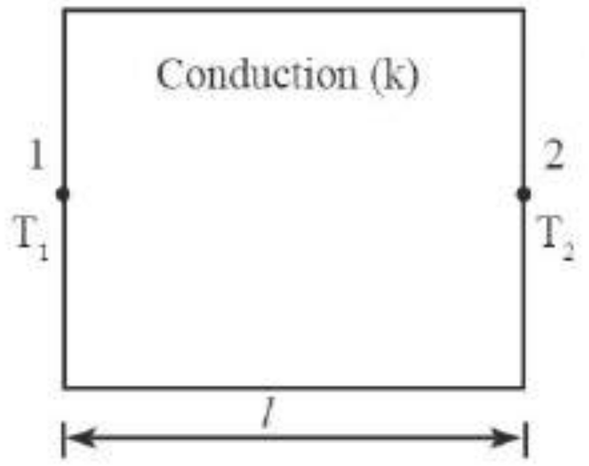
Thermal stiffness matrix

$[K_c]$ – Stiffness matrix in case of heat conduction

$$= \frac{KA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$\{T\}$ – Nodal temperature vector

$$= \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} \text{ for a two noded element}$$



FE Equation for 1D heat conduction and Convection element

$$[K]\{T\}=\{F\}$$

Stiffness Matrix in Case of a 1D Fin Pro

Thermal Stiffness matrix, $[K] = [K_c] + [K_{he}]$

$$[k]=\frac{KA}{l}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}+hA\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$$

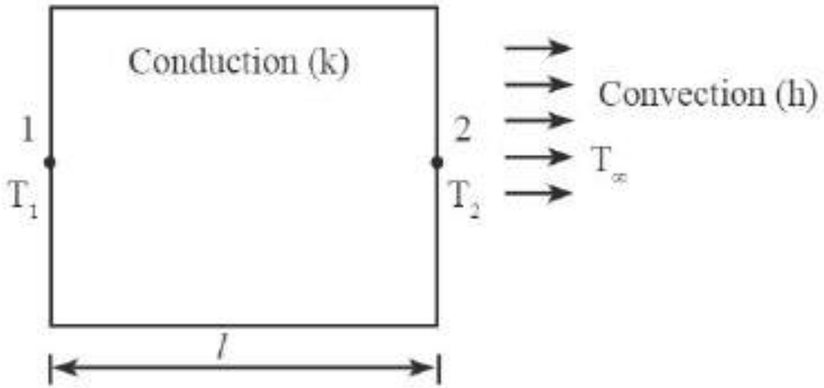


Figure: Element with Free End Convection

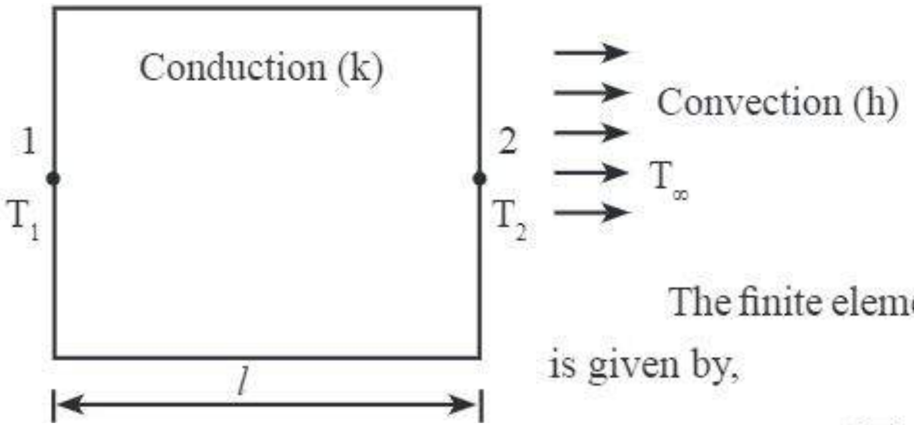
$$[K_c] = \frac{AK}{l}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$ – Conduction matrix

(Free end convection matrix)

$$[K_{he}] = hA\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$$ – Convection matrix

$$hT_\infty A\begin{Bmatrix}0\\ 1\end{Bmatrix}$$ – Thermal load matrix

A	⟶	Area of the wall	- m^2
k	⟶	Thermal conductivity of wall	- W/mK
l	⟶	Length of the wall	- m
h	⟶	Heat transfer coefficient	- W/m^2K
T_∞	⟶	Atmospheric air temperature	- K



The finite element equation for in one dimension element is given by,

$$\{F\} = [K]\{T\} \quad \text{... (1)}$$

In case of conduction, stiffness matrix is given by,

$$[K_c] = \frac{AK}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad \text{... (2)}$$

Where,

K – Thermal conductivity of the element material, W/m-K

A – Area of the element, m^2

l – Length of the element, m

In case of convection (end), stiffness matrix is given by,

$$[K_{he}] = \iint_A h[N]^T [N] dA \qquad \dots (3)$$

N – Shape functions

h – Heat transfer coefficient, W/m^{2}K

$$[N] = [N_1 \quad N_2]$$

$$= \begin{bmatrix} \frac{l-x}{l} & \frac{x}{l} \end{bmatrix}$$

at $x = l$, at the end of element

$$[N] = [N_1 \quad N_2] = [0 \quad 1]$$

$$[N]^T = \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

From equation (3),

$$[K_{he}] = \iint_A h.[N]^T [N].dA$$

$$= \iint_A h.\begin{bmatrix} 0 \\ 1 \end{bmatrix} [0 \quad 1] dA$$

$$= h \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \iint_A dA$$

$$[K_{he}] = hA \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Stiffness matrix,

$$[K] = [K_c] + [K_{he}]$$

$$[K] = \frac{AK}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + hA\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Convective force vector at the free end,

$$\{F_{he}\} = hT_{\infty}A\begin{bmatrix} N_1 \\ N_2 \end{bmatrix}$$

Where,

T_{∞} – Fluid temperature

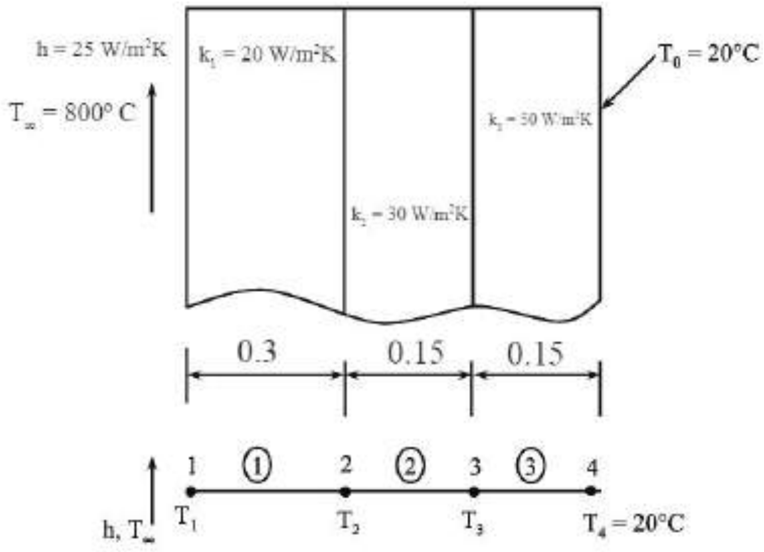
$$\{F_{he}\} = hT_{\infty}A\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

From equation (1),

$$\{F_{he}\} = [K]\ \{T\}$$

$$hT_{\infty}A\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \left[\frac{AK}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + hA\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right]\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

A composite slab consists of three materials of different conductivities is 20 W/mK, 30 W/mK and 50 W/mK of thickness 0.3 m, 0.15 m and 0.15 m respectively. The outer surface is 20°C and the inner surface is exposed to the convective heat transfer coefficient of 25 W/m²K at 800°C. Determine the temperature distribution within the wall.



Given that,

For material-1,

Thermal conductivity, $K_1 = 20$ W/mK

Thickness, $L_1 = 0.3$ m

For material-2,

Thermal conductivity, $K_2 = 30$ W/mK

Thickness, $L_2 = 0.15$ m

For material-3,

Thermal conductivity, $K_3 = 50$ W/mK

Thickness, $L_3 = 0.15$ m

Temperature of outer surface, $T_o = 20$°C.

Convective heat transfer coefficient, $h = 25$ W/m²K

Ambient temperature, $T_\infty = 800$°C

Element (1): It is subjected to both conduction and convection. Convection is only from left side. Therefore, finite element equation for element (1) can be written as,

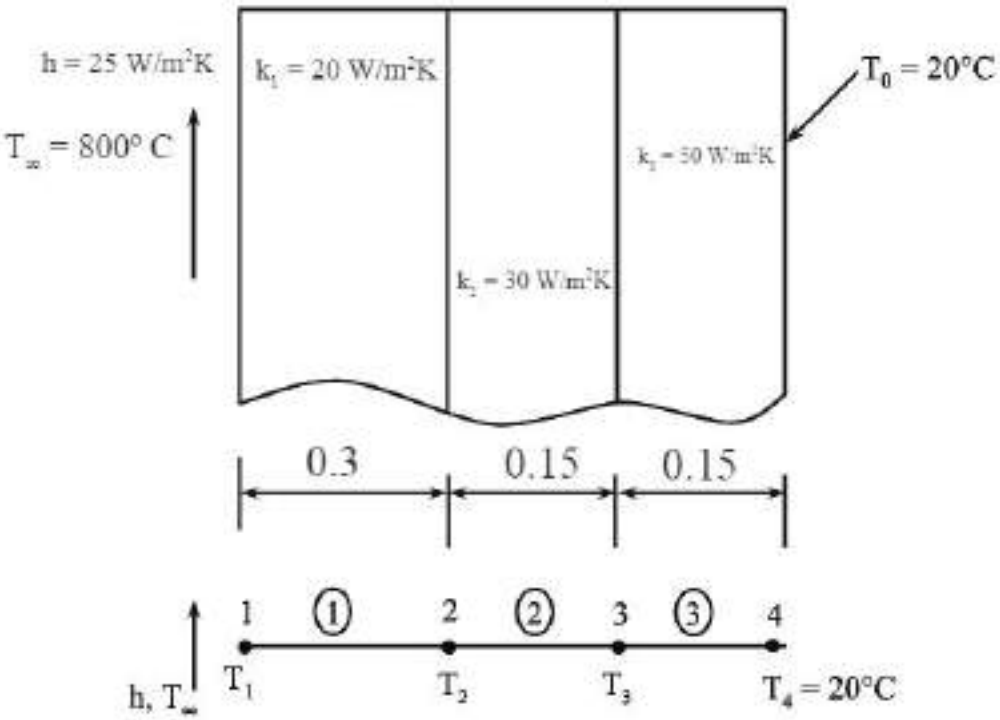
$$[K_1] = \frac{A_1 K_1}{l_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + hA\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Assume, $A_1 = 1\ \text{m}^2$

$$= \frac{1 \times 20}{0.3}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + 25 \times 1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 66.66 & -66.66 \\ -66.66 & 66.66 \end{bmatrix} + \begin{bmatrix} 25 & 0 \\ 0 & 0 \end{bmatrix}$$

$$[K_1] = \begin{array}{c} \begin{matrix} \quad 1 & \qquad 2 \quad \end{matrix} \\ \begin{bmatrix} 91.66 & -66.66 \\ -66.66 & 66.66 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \end{array}$$

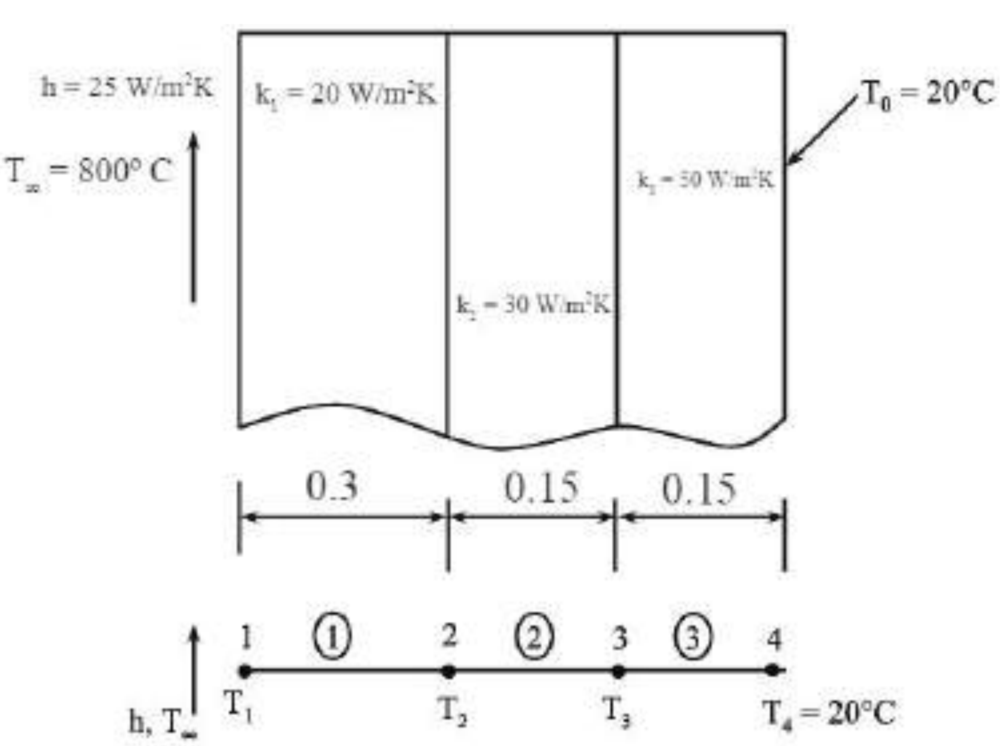


Since, on the left end convection takes place, load vector on the left is given by,

$$\{F_{he}\} = hT_{\infty}A\begin{Bmatrix}1\\0\end{Bmatrix}$$

1

$$\{F_{he}\} = \begin{array}{c}\begin{Bmatrix}hT_{\infty}A\\0\end{Bmatrix}\end{array}\begin{array}{c}1\\2\end{array}$$



Element (2)

$$[K_2] = \frac{A_2 K_2}{l_2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{1 \times 30}{0.15}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

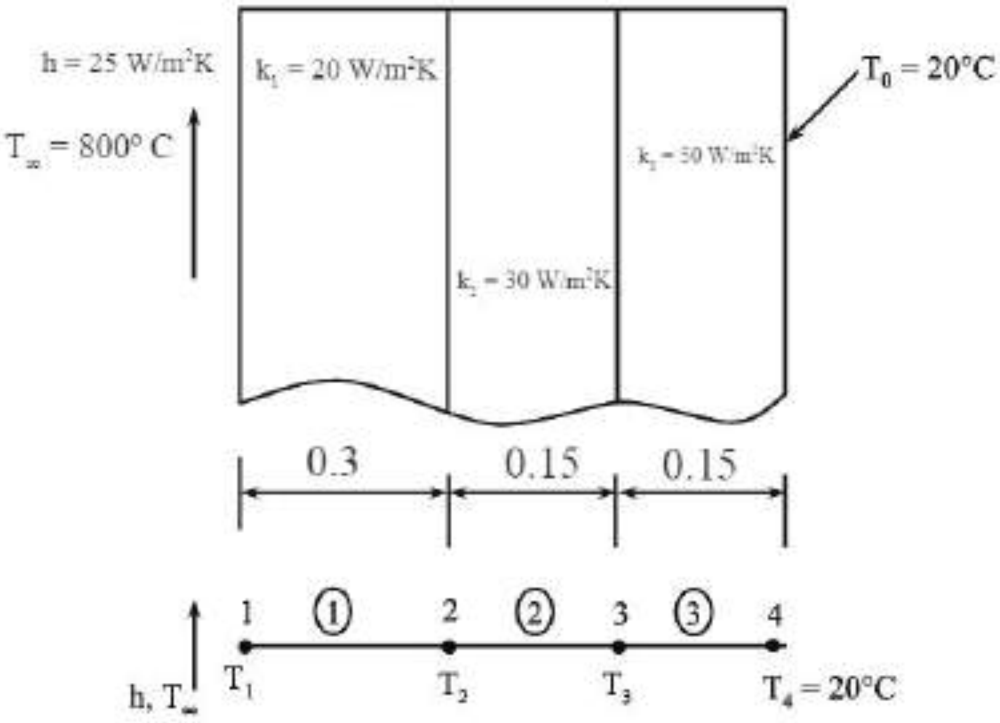
$$[K_2] = \begin{array}{c} \begin{array}{cc} 2 & \quad 3 \end{array} \\ \begin{bmatrix} 200 & -200 \\ -200 & 200 \end{bmatrix} \end{array} \begin{array}{c} \\ 2 \\ 3 \end{array}$$

Element (3)

$$[K_3] = \frac{A_3 K_3}{l_3}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{1 \times 50}{0.15}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[K_2] = \begin{array}{c} \begin{array}{cc} 3 & \quad 4 \end{array} \\ \begin{bmatrix} 333.333 & -333.333 \\ -333.333 & 333.333 \end{bmatrix} \end{array} \begin{array}{c} \\ 3 \\ 4 \end{array}$$



Global stiffness matrix,

$$[K] = [K_1] + [K_2] + [K_3]$$

$$[K] = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 91.66 & -66.66 & 0 & 0 \\ -66.66 & 66.66 + 200 & -200 & 0 \\ 0 & -200 & 200 + 333.33 & -333.33 \\ 0 & 0 & -333.33 & 333.33 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}$$

$$[\mathrm{K}] = \begin{bmatrix} 91.66 & -66.66 & 0 & 0 \\ -66.66 & 266.66 & -200 & 0 \\ 0 & -200 & 533.33 & -333.33 \\ 0 & 0 & -333.33 & 333.33 \end{bmatrix}$$

Finite element equation for the given composite slab is given by,

$$[K]\{T\} = \{F\}$$

$$\begin{bmatrix} 91.66 & -66.66 & 0 & 0 \\ -66.66 & 266.66 & -200 & 0 \\ 0 & -200 & 533.33 & -333.33 \\ 0 & 0 & -333.33 & 333.33 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} hT_\infty A \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$hT_\infty A = 25 \times 800 \times 1$$

$$hT_\infty A = 20000$$

$$\begin{bmatrix} 91.66 & -66.66 & 0 & 0 \\ -66.66 & 266.66 & -200 & 0 \\ 0 & -200 & 533.33 & -333.33 \\ 0 & 0 & -333.33 & 333.33 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} 20000 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$T_1 = 304.77^{\circ}\mathrm{C}$$

$$T_2 = 119.04^{\circ}\mathrm{C}$$

$$T_3 = 57.142^{\circ}\mathrm{C}$$

$$T_4 = 20^{\circ}\mathrm{C}$$

Temperature distribution within the wall is given by,

$$\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} 304.77 \\ 119.04 \\ 57.142 \\ 20 \end{Bmatrix}$$

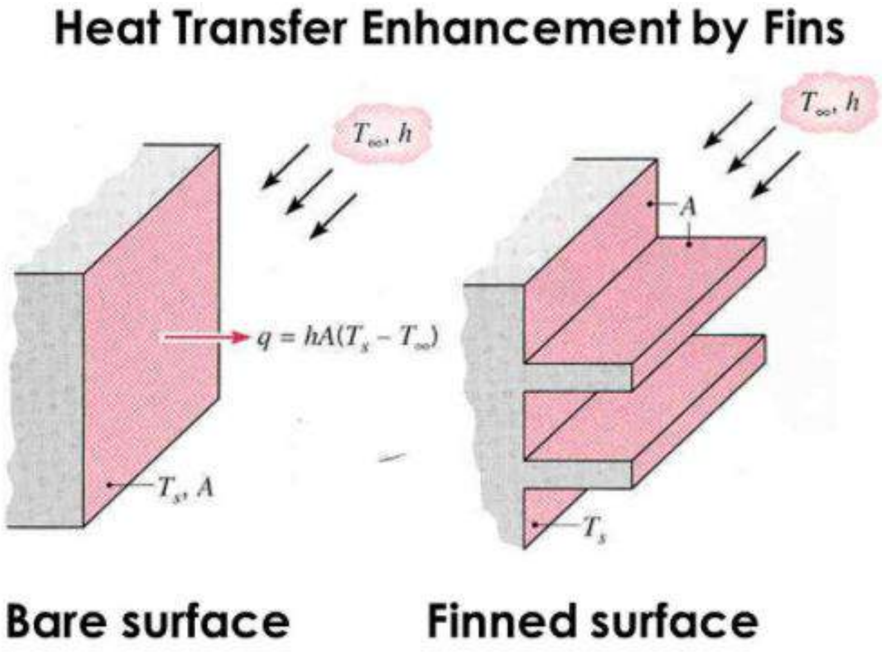
One-dimensional Heat Transfer: When the temperature and heat transfer in a system vary only in one direction, it is known as one-dimensional heat transfer. In this, the variation of temperature in other two directions is negligible.

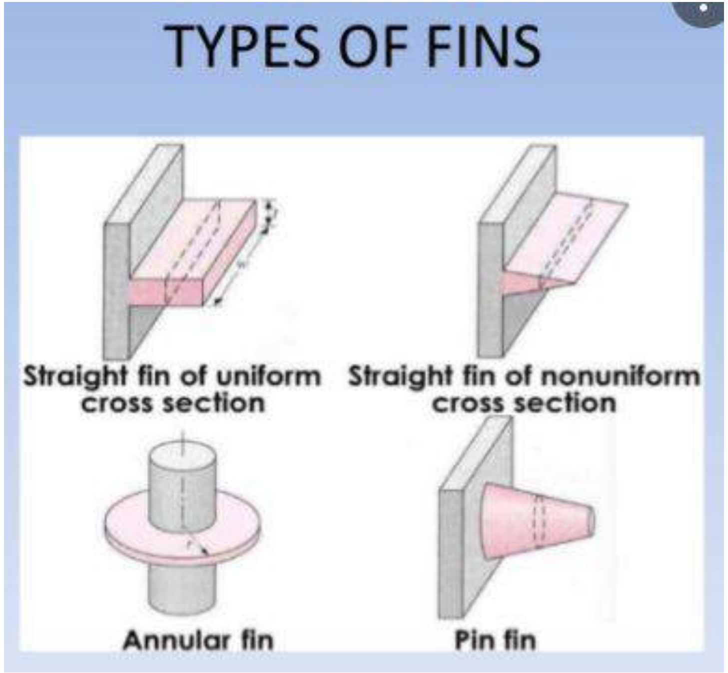
Example: Heat transfer from a glass window is considered to be one-dimensional, as the heat transfer takes place in one direction, whereas in other directions it is zero.

One dimensional analysis of a fin

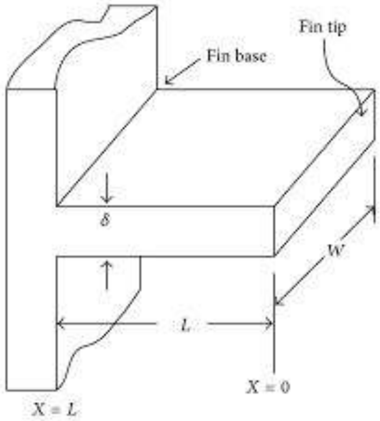
Fin (extended surface)

Fin is a metallic strip of rectangular shape or circular shape and integral with the surface, through which heat is to be transferred. Fin increases the surface area of heat transfer and also, known as extended surface. They are used on engine cylinders, heat exchanger pipes, etc.





A **fin** subjected to conduction and convection



$$[K_c] = \frac{Ak}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[K_h] = \frac{hPl}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$[K] = [K_c] + [K_h]$$

$$[K]\{T\} = \{F\}$$

$$\left[\frac{Ak}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{hPl}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right]\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \frac{QAl + PhT_\infty l}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$A = Length * Thickenss = l * t$$

$$P = 2 * l \ (Approximately)$$

A	→	Area of the fin	- m^2
P	→	Perimeter of the fin	- m
k	→	Thermal conductivity of fin	- W/mK
l	→	Length of the fin	- m
h	→	Heat transfer coefficient	- W/m^2K
T_∞	→	Atmospheric air temperature	- K
Q	→	Heat Generation	- W

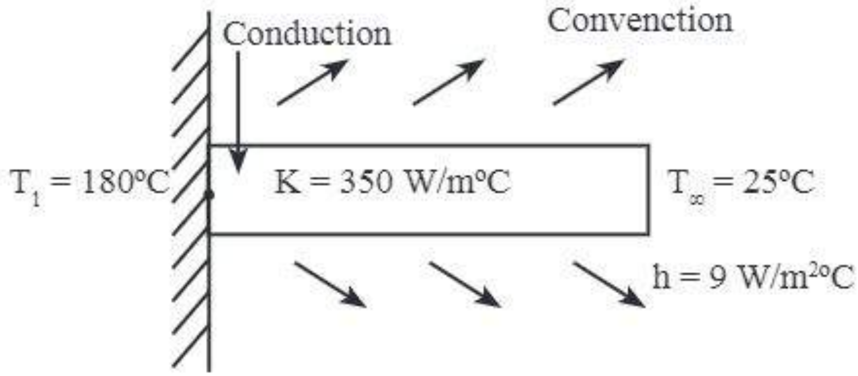
If at free end convection exist

(Free end convection matrix)

$$[K_{he}] = hA\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} - \text{Convection matrix}$$

$$\boxed{K = K_c + K_h + K_{he}}$$

A metallic fin 20 mm wide and 4 mm thick is attached to a furnace whose wall temperature is 180°C. The length of the fin is 120 mm. If the thermal conductivity of the material of the fin is 120mm. If the thermal conductivity of the material of the fin is 350 W/m°C and convection coefficient is 9 W/m²°C, determine the temperature distribution assuming that the tip of the fin is open to the atmosphere and that the ambient temperature is 25°C.



A fin (length 120mm, 20mm wide and 4mm thick) is attached to a furnace wall temperature of 180 C. Determine the temperature at the midpoint of the fin assuming the tip of the fin is open to atmosphere, which is at 25 C(take fin's conductivity 350 W/mK And convection coefficient of atmosphere 9 W/m^2K)

Give that,

Width of the fin, $w = 20$ mm $= 0.02$ m

Thickness of the fin, $t = 4$ mm $= 4 \times 10^{-3}$ m

Wall temperature, $T_1 = 180$°C

Length of the fin, $L = 120$ mm $= 0.12$ m

Thermal conductivity, $K = 350$ W/m°C

Convection coefficient, $h = 9$ W/m²°C

Ambient temperature, $T_\infty = 25$°C

The fin is divided into two equal elements.

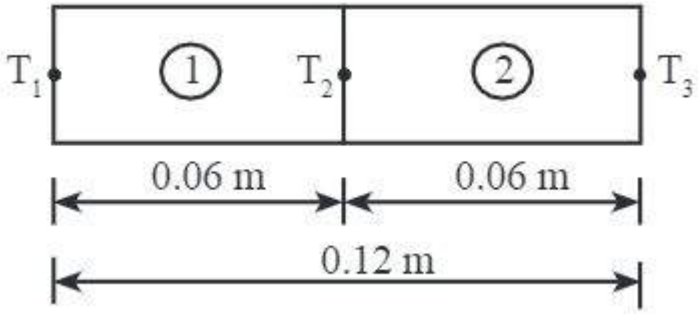


Figure (2): F.E Modal

Give that,

Width of the fin, w = 20 mm = 0.02 m

Thickness of the fin, t = 4 mm = 4×10^{-3} m

Wall temperature, T_1 = 180°C

Length of the fin, L = 120 mm = 0.12 m

Thermal conductivity, K = 350 W/m°C

Convection coefficient, h = 9 W/m^2°C

Ambient temperature, T_∞ = 25°C

Element (1)

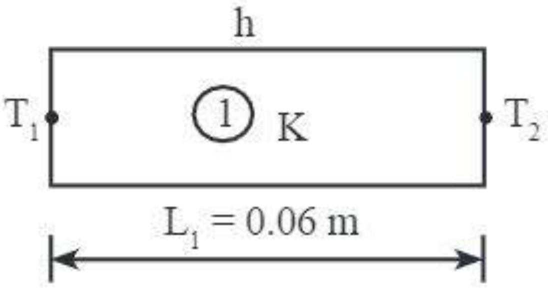


Figure (3): Element (1)

Stiffness matrix for element (1).

$$[K] = \frac{A_1 K_1}{l_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{hpl_1}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$A_1 = \mathrm{W} \times t = 0.02 \times 4 \times 10^{-3} = 8 \times 10^{-5}\ \mathrm{m}^2$$

$$P = 2(\mathrm{W} + t) = 2(0.02 + (4 \times 10^{-3}))$$

$$P = 0.012\ \mathrm{m}$$

Force vector for element (1).

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{QA_1 l_1 + phT_\infty l_1}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

Since, 'Q' is not given in the problem, neglect the term $\left(\frac{QA_1 l_1}{2}\right)$.

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{phT_\infty l_1}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

Stiffness matrix for element (1).

$$[K] = \frac{A_1 K_1}{l_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{hpl_1}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= \frac{8\times 10^{-5}\times 350}{0.06}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{9\times 0.012\times 0.06}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= 0.467\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \left(1.08\times 10^{-3}\right)\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0.467 & -0.467 \\ -0.467 & 0.467 \end{bmatrix} + \begin{bmatrix} 2.16\times 10^{-3} & 1.08\times 10^{-3} \\ 1.08\times 10^{-3} & 2.16\times 10^{-3} \end{bmatrix}$$

$$[K_1] = \begin{array}{c} \begin{array}{cc} \quad 1 & \qquad\quad 2 \end{array} \\ \begin{bmatrix} 0.46916 & -0.46592 \\ -0.46592 & 0.46916 \end{bmatrix} \end{array}\begin{array}{c} \\ 1 \\ 2 \end{array}$$

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \frac{phT_\infty l_1}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$= \frac{0.012\times 9\times 25\times 0.06}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \begin{Bmatrix} 0.081 \\ 0.081 \end{Bmatrix}$$

Finite element equation for element (1).

$$\begin{bmatrix} 0.46916 & -0.46592 \\ -0.46592 & 0.46916 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$\begin{array}{c c} \begin{array}{cc} \quad 1 \quad & \quad 2 \quad \end{array} & \\ \begin{bmatrix} 0.46916 & -0.46592 \\ -0.46592 & 0.46916 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \end{array} \end{array} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} 0.081 \\ 0.081 \end{Bmatrix}$$

Element (2)

Since all the parameters and properties are same finite element equation for element (2) is given by,

$$\begin{bmatrix} 0.46916 & -0.46592 \\ -0.46592 & 0.46916 \end{bmatrix} \begin{Bmatrix} T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix}$$

$$\begin{array}{c c} \begin{array}{cc} \quad 2 \quad & \quad 3 \quad \end{array} & \\ \begin{bmatrix} 0.46916 & -0.46592 \\ -0.46592 & 0.46916 \end{bmatrix} & \begin{array}{c} 2 \\ 3 \end{array} \end{array} \begin{Bmatrix} T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 0.081 \\ 0.081 \end{Bmatrix}$$

$$\begin{array}{cc} \begin{array}{cc} 1 & 2 \end{array} & \\ \begin{bmatrix} 0.46916 & -0.46592 \\ -0.46592 & 0.46916 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \end{array} \end{array} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} 0.081 \\ 0.081 \end{Bmatrix}$$

$$\begin{array}{cc} \begin{array}{cc} 2 & 3 \end{array} & \\ \begin{bmatrix} 0.46916 & -0.46592 \\ -0.46592 & 0.46916 \end{bmatrix} & \begin{array}{c} 2 \\ 3 \end{array} \end{array} \begin{Bmatrix} T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 0.081 \\ 0.081 \end{Bmatrix}$$

Global finite element equation

$$\begin{array}{cc} \begin{array}{ccc} 1 & 2 & 3 \end{array} & \\ \begin{bmatrix} 0.46916 & -0.46592 & 0 \\ -0.46592 & 0.46916+0.46916 & -0.46592 \\ 0 & -0.46592 & 0.46916 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \end{array} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 0.081 \\ 0.081+0.081 \\ 0.081 \end{Bmatrix}$$

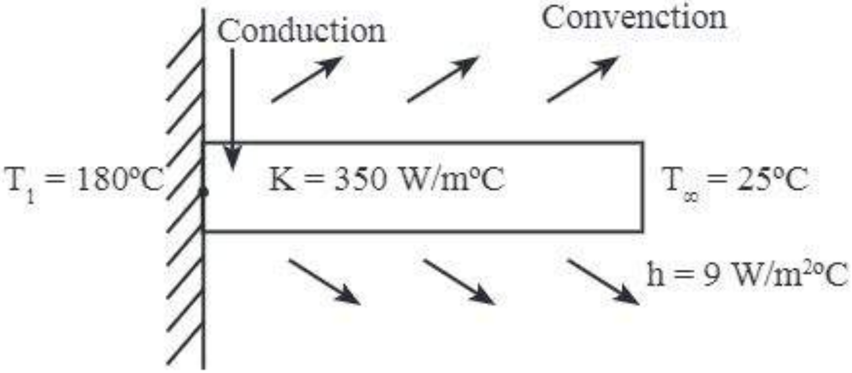
$$\begin{bmatrix} 0.46916 & -0.46592 & 0 \\ -0.46592 & 0.93832 & -0.46591 \\ 0 & -0.46592 & 0.46916 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 0.081 \\ 0.162 \\ 0.081 \end{Bmatrix}$$

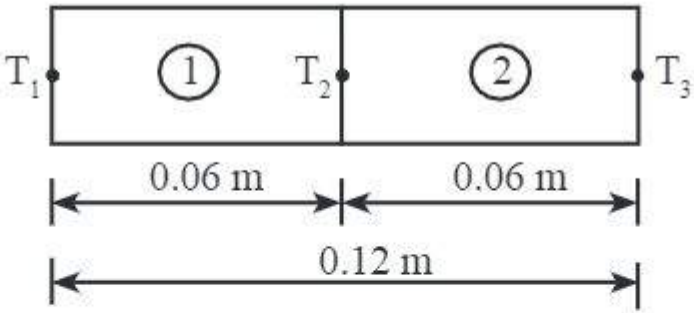
T1=180 C

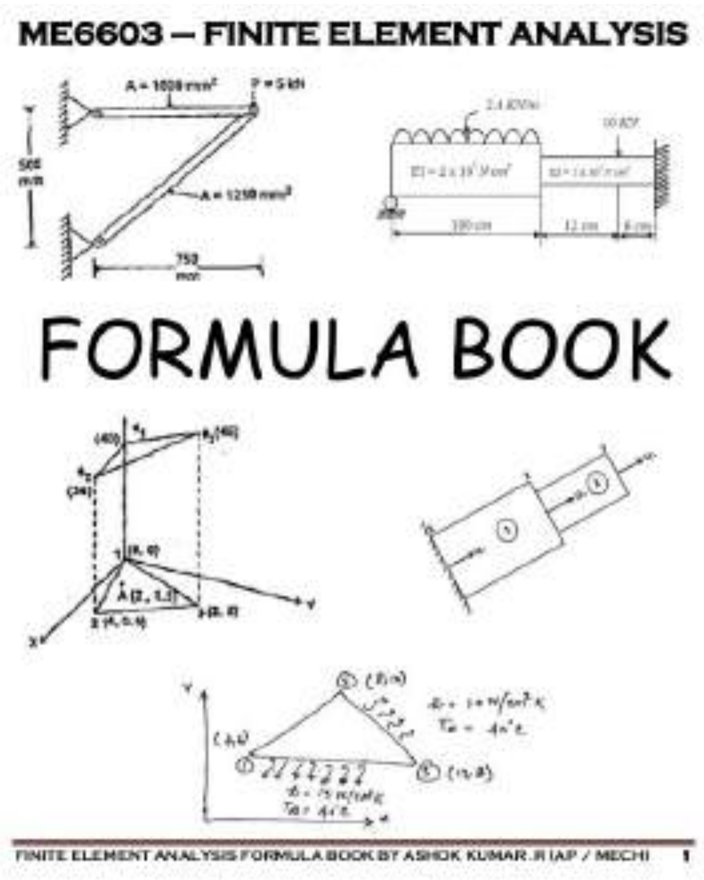
$$\begin{bmatrix} \cancel{0.46916} & \cancel{-0.46592} & \cancel{0} \\ -0.46592 & 0.93832 & -0.46591 \\ 0 & -0.46592 & 0.46916 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} \cancel{0.081} \\ 0.162 \\ 0.081 \end{Bmatrix}$$

Writing the 2nd and 3rd rows

$$\left.\begin{aligned} \mathbf{-0.46592\ T1 + 0.93832\ T2 - 0.46591\ T3 = 0.162} \\ \mathbf{-\ 0.46592\ T2 + 0.46916\ T3 = 0.081} \end{aligned}\right\} \qquad \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 180 \\ 176.83 \\ 175.78 \end{Bmatrix}$$



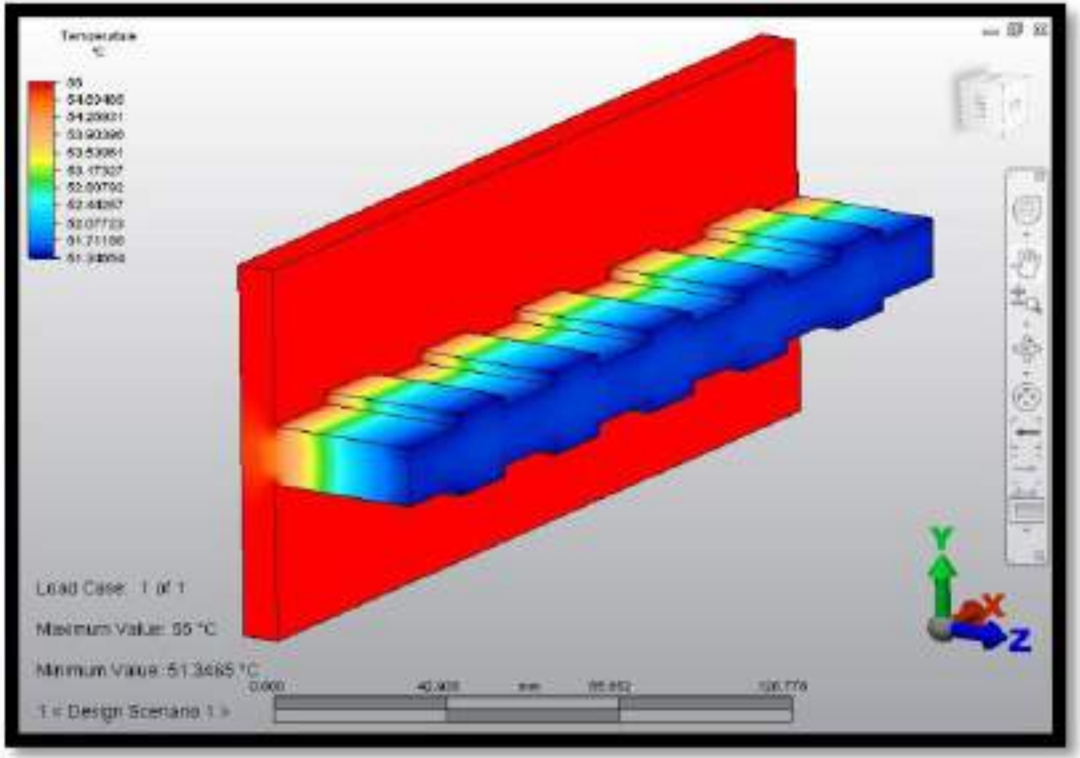


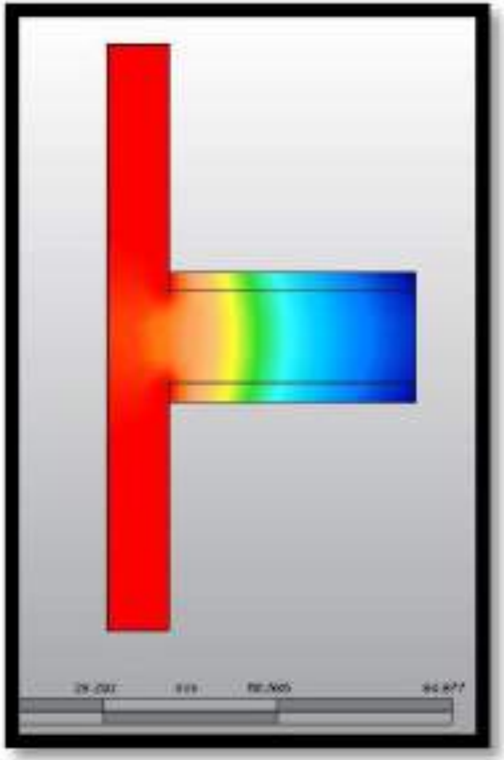

ME6603 – FINITE ELEMENT ANALYSIS

FORMULA BOOK

FINITE ELEMENT ANALYSIS FORMULA BOOK BY ASHOK KUMAR .R (AP / MECH) 1

https://www.slideshare.net/ASHOKKUMAR27088700/me6603-finite-element-analysis-formula-book


Temperature
°C
Load Case: 1 of 1
Maximum Value: 55 °C
Minimum Value: 51.3465 °C
1 < Design Scenario 1 >
Y
X
Z



Explain about the use of ANSYS in FEA.

Ans: ANSYS is a finite element modeling package and design analysis tool which is used to solve different problems of engineering based on structural analysis, thermal analysis, CFD analysis, etc.

The documentation of a product designed by ANSYS software consists of commands reference, operations guide, modeling guide, element reference, etc.

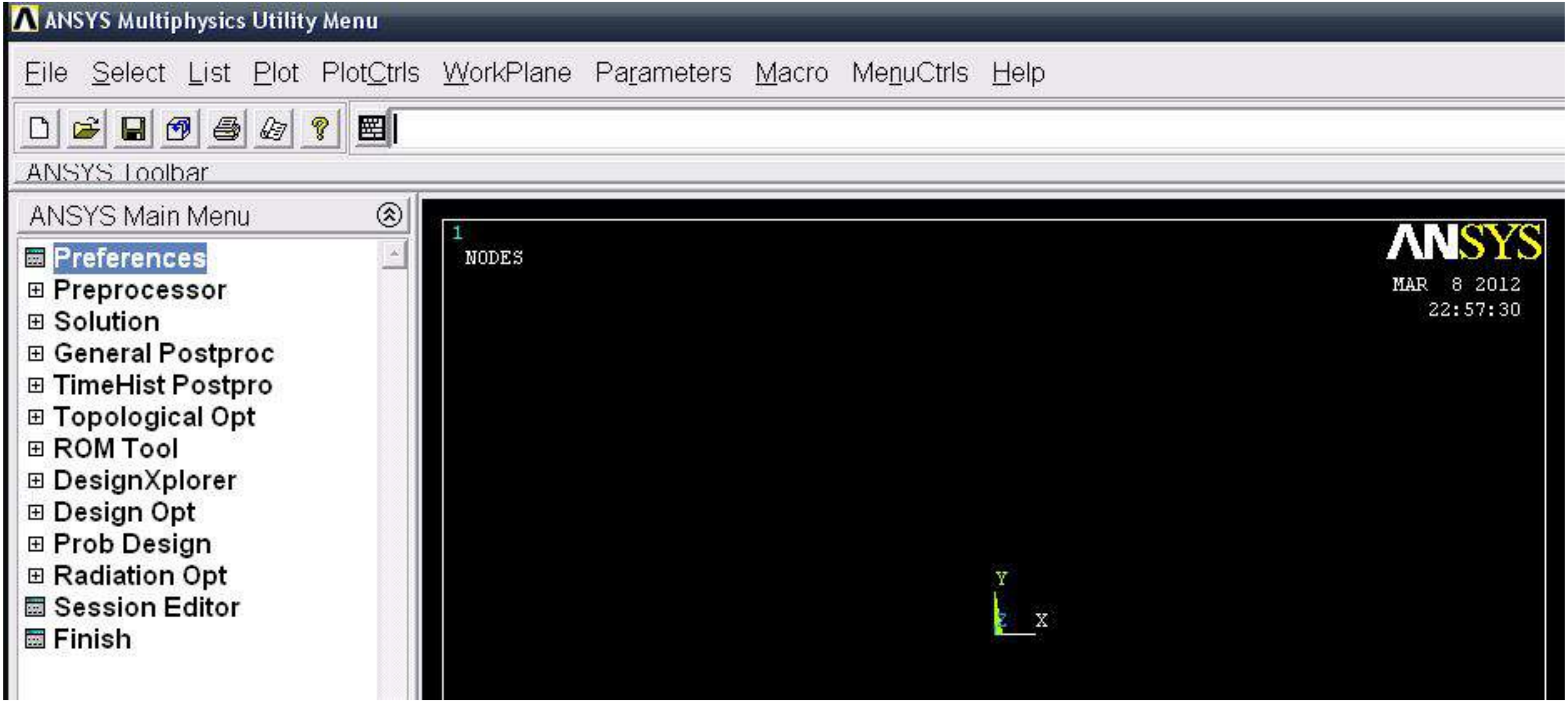
Solving the problem using ANSYS is carried out in three stages namely,

1. Preprocessing stage
2. Solution stage
3. Postprocessing stage.

ANSYS – Analysis of Systems

ANSYS®

Full Form of APDL is
ANSYS Parametric Design Language



Basic overview: https://www.youtube.com/watch?v=ePA9csthHNM
Tutorials: https://sites.ualberta.ca/~wmoussa/AnsysTutorial/BT/BT.html

1. **Preprocessing Stage**

In this stage, the problem is described or stated clearly by the following steps,

(i) Defining the key points, lines and areas of elements.

(ii) Defining the element type which includes the elements shape, dimensions, degrees of freedom, etc.

(iii) Defining the material properties like Young's modulus, Poisson's ratio, thermal conductivity, etc.

(iv) Stating the mesh lines, areas or volumes as per the requirement.

2. Solution Stage

After the preprocessing stage, the next stage is solution stage, which includes specifying loads, constraints and obtaining the solution. The sequential steps are as follows,

(i) Stating the loads which may include pressure or points.

(ii) Adding the constraints, either translational or rotational.

(iii) Solving the equations, which are associated with the above steps.